Problem for 2005 December

Proposed by Jennifer Jean Jurca

Suppose that n is a positive integer, and n married couples attend a party. Naturally there is some handshaking going on, although no person shakes hands with herself or himself, or with her or his spouse. One woman decides to find out how many hands each person has shaken, so she has each person-excluding herself-write down the number of hands she or he has shaken. Unfortunately, no person wrote down a name. However, she observes that each number is a nonnegative integer strictly less than $2n-1$, and no number occurs more than once. She asks how many hands her husband has shaken.

Can you help her?

Solution by the proposer

Her husband shook $n-1$ hands.

This is immediate in the case $n=1$, and we use induction to show the same if $1 < n$. So suppose that $1 < n$, and that at any party as in the problem statement with $n-1$ couples the husband shook $n-2$ hands. For convenience let Q be the woman asking the question, and for each person P at the party, suppose that P′ is the spouse of P; thus Q asks how many hands Q′ has shaken. Let X be the person who shook the greatest number of hands; *i.e.*, X is the person who shook $2n-2$ hands (and wrote $2n-2$). Thus X is not Q, since X wrote something, but Q did not. We claim that X′ shook 0 hands (and wrote 0). For there were 2n persons at the party but X shook the hand of neither X nor X′, so must have shaken the hand of each of the $2n-2$ persons other than X or X′; it follows that each person other than X or X′ shook at least one hand, so each person other than X or X′ (and other than Q) wrote a number different from 0. Hence the person who wrote 0 must be X′.

Now suppose that X and X′ are excused from the party, they take the numbers (0 and $2n-2$) which they have written, and each other number is decremented by 1 to the value it would have been had X and X′ not been in attendance but all other hand-shaking had been as before. Thus the 1 becomes 0, the 2 becomes 1, *etc.* The situation now is precisely what it would have been if only $n-1$ couples had attended the party originally. Therefore, by the inductive hypothesis, Q′ shook $n-2$ hands of the $2n-2$ remaining persons (and wrote $n-2$). Now recalling X and X′ we see that Q′ shook $n-1$ hands, as asserted.

Also solved by Massoud Malek and Victor Manjarrez.