Problem for 2006 January

Communicated by Dan Jurca

Evaluate the following sums, if they exist.

$$\sum_{n=1}^{\infty} (\arctan(n) - \arctan(n-1))$$

$$\sum_{n=1}^{\infty} \arctan \frac{1}{n^2 - n + 1}$$

Solution by Dan Jurca

We show each series sums to $\pi/2$.

First we observe that

$$1 \le N \Rightarrow \sum_{n=1}^{N} (\arctan(n) - \arctan(n-1)) = \arctan(N).$$

This is obvious if N=1, since arctan(0)=0, and for $2 \le N$ follows by an easy induction on N. Therefore

$$\sum_{n=1}^{\infty} (\arctan(n) - \arctan(n-1)) = \lim_{N\to\infty} \sum_{n=1}^{N} (\arctan(n) - \arctan(n-1))$$

$$= \lim_{N\to\infty} \arctan(N)$$

$$=\pi/2.$$

Next from the formula tan(A–B)=(tanA–tanB)/(1+tanAtanB), with A=arctan(a) and B=arctan(b) we have

$$\arctan(a)-\arctan(b)=\arctan\frac{a-b}{1+ab},$$

whence with a=n and b=n–1 we find

$$\arctan(n)-\arctan(n-1) = \arctan\frac{n-(n-1)}{1+n(n-1)}$$

$$= \arctan\frac{1}{n^2-n+1},$$

so that the series sum to the same value, since the terms are the same.

Also solved by Massoud Malek and Gagan Sekhon