Problem for 2006 February

Proposed by Dan Jurca

According to the 2005 December issue of the Canadian mathematics journal *Crux Mathematicorum* the following problem appeared in the 2005 Maritime Mathematics Competition.

> Three students play a game with the understanding that the loser is to double the money of each of the other two. After three games, each has lost once and each has \$24. How much did each student have to start?

a.
Solve the problem above.
b.
Generalize as follows. Suppose there are n players, say the players are P_1, P_2, …, P_n, and the notation is such that player P_i loses exactly once, the i-th game. Again suppose that the loser of a game doubles the money of each of the other players, that the n players play n games, and that after these n games each player has an amount A. For $1 \le i \le n$ and $0 \le j \le n$ let a_{ij} be the amount had by player P_i after j games have been played. Thus player P_i begins with a_{i0}, and for each i, $1 \le i \le n$, $a_{in}=A$.
Determine a_{ij}.

Solution by Dan Jurca

By the conditions in the statement of the problem one has the following.

$$1 \le i \le n \text{ and } 1 \le j \le n \;\Rightarrow a_{ij} = \begin{cases} 2a_{i,j-1} & \text{if } j \ne i, \\ a_{i,j-1} - \displaystyle\sum_{k=1,\, k \ne i}^{n} a_{k,j-1} & \text{if } i=j. \end{cases}$$

Since after n games the total amount of money had by all the players equals nA, it follows that

$$0 \le j \le n \Rightarrow \sum^{n} a_{kj} = nA, \quad \text{so that}$$

k=1

$$0 \le j \le n \Rightarrow \sum_{k=1,\, k \neq i}^{n} a_{kj} = nA - a_{ij},$$

and one can restate the condition above as follows.

$$1 \le i \le n \text{ and } 1 \le j \le n \Rightarrow a_{ij} = \begin{cases} 2a_{i,j-1} & \text{if } j \neq i, \\ 2a_{i,i-1} - nA & \text{if } i = j. \end{cases}$$

Now there exists a $n \times (n+1)$ matrix with rows $1,2,\ldots,n$ and columns $0,1,\ldots,n$ containing a_{ij} in row i and column j as follows.

$$\begin{pmatrix} a_{10} & 2a_{10}-nA & 4a_{10}-2nA & 8a_{10}-4nA & \ldots & 2^n a_{10}-2^{n-1}nA \\ a_{20} & 2a_{20} & 4a_{20}-nA & 8a_{20}-2nA & \ldots & 2^n a_{20}-2^{n-2}nA \\ a_{30} & 2a_{30} & 4a_{30} & 8a_{30}-nA & \ldots & 2^n a_{30}-2^{n-3}nA \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0} & 2a_{n0} & 4a_{n0} & 8a_{n0} & \ldots & 2^n a_{n0}-nA \end{pmatrix}$$

It follows by inspection that

$$1 \le i \le n \text{ and } 0 \le j \le n \Rightarrow a_{ij} = \begin{cases} 2^j a_{i0} & \text{if } j < i, \\ 2^j a_{i0} - 2^{j-i} nA & \text{if } i \le j. \end{cases}$$

Since $1 \le i \le n \Rightarrow a_{in}=A$, we find

$$2^n a_{i0} - 2^{n-i} nA = A, \quad \text{whence}$$

$$a_{i0} = \frac{A + 2^{n-i} nA}{2^n}$$

$$= \frac{2^{n-i} n + 1}{2^n} A, \quad \text{so that, finally}$$

$$1 \le i \le n \text{ and } 0 \le j \le n \Rightarrow a_{ij} = \begin{cases} \dfrac{2^{n-i+j} n + 2^j}{2^n} A & \text{if } j < i, \\ \dfrac{2^j}{2^n} A & \text{if } i \le j, \end{cases}$$

which one can verify by induction.

In the special case n=3, A=\$24, we have a_{10}=\$39, a_{20}=\$21, and a_{30}=\$12.

Also solved by Massoud Malek
