Problem for 2006 March

Proposed by Dan Jurca

Find a non-constant continuous function $f:[0,\infty)\to\mathbf{R}$ such that for each x, $0 \le x$, if S_x is the surface obtained by revolving (about the x-axis) the graph of f from the y-axis to the vertical line through the point (x,0), then the volume enclosed by S_x (with ends at 0 and x) equals the surface area of S_x.

Remark. Here the surface area of S_x does not include the areas of the circles at the ends.

Solution by the proposer

Using the formulas for volume and surface area the conditions in the problem require

$$0 \le x \Rightarrow \pi \int_0^x f^2 = 2\pi \int_0^x f \sqrt{1+(f')^2}$$

so that by equating the derivatives of the two sides of the equation we have (by the fundamental theorem of calculus)

$$0 \le x \Rightarrow \pi(f(x))^2 = 2\pi f(x) \sqrt{1+(f'(x))^2} \,,$$

which holds if f satisfies

$$0 \le x \Rightarrow f(x) = 2 \sqrt{1+(f'(x))^2} \,.$$

Writing $y=f(x)$ we have the differential equation $y=2\sqrt{1+(y')^2}$, or $y^2=4(1+(y')^2)$, so that we have the separable equation

$$\frac{dy}{\sqrt{y^2-4}} = \frac{dx}{2}.$$

Hence $\ln|y+\sqrt{y^2-4}|=x/2+C$, some C. Setting $y(0)=2$ we find $C=\ln 2$, so that

$$\begin{aligned}
\ln|y+\sqrt{y^2-4}| &= x/2+\ln 2 \\
&= \ln 2e^{x/2}, \quad \text{so that} \\
y+\sqrt{y^2-4} &= 2e^{x/2}, \quad \text{whence, solving for } y, \\
y &= e^{x/2}+e^{-x/2} \\
&= 2\cosh(x/2).
\end{aligned}$$

One easily checks that $f(x)=2\cosh(x/2)$ satisfies the conditions in the problem. Other solutions exist.

Also solved by Massoud Malek, Bill Nico and John Sayer