Problem for 2006 April

Proposed by Dan Jurca

The following problem is a variation of problem 11209 which appears in the 2006 March issue of *The American Mathematical Monthly*.

Show that there do not exist positive real numbers x_1, x_2, and x_3 which satisfy the following system of equations.

$$(x_1+x_2+x_3)^{x_1} = 6/7$$
$$(x_1+x_2+x_3)^{x_2} = 7/8$$
$$(x_1+x_2+x_3)^{x_3} = 8/9$$

Solution by the proposer

Assuming that there exist x_1, x_2, and x_3 which satisfy the given system we obtain a contradiction as follows. With $X=x_1+x_2+x_3$ we find the product of the equations yields the equation

$$X^{x_1}\times X^{x_2}\times X^{x_3} = 6/7\times 7/8\times 8/9, \quad \text{so that}$$
$$X^{x_1+x_2+x_3} = 6/9, \text{or}$$
$$X^X = 2/3.$$

However, if $\varphi:(0,\infty)\to\mathbf{R}$ by $\varphi(x)=x^x$, we find

$$\varphi'(x) = x^x(1+\ln x)$$
$$= 0 \text{ if and only if } x=1/e,$$

and that $\varphi_{min}=\varphi(1/e)=(1/e)^{1/e}$. Since $2/3 < (1/e)^{1/e} \approx 0.6922$, there exists no real X such that $X^X=2/3$; hence there exists no solution of the system.

Also solved by Massoud Malek, Craig Prescott, John M. Sayer, and Nathan Speed