Problem for 2006 May

Communicated by Dan Jurca

a.

Show that there exists a bijection $\mathbf{R}^{\mathbf{N}} \to \mathbf{R}$; *i.e.*, from the set of sequences of real numbers to the set of real numbers.
Here $\mathbf{N}=\{0,1,2,\ldots\}$, the set of natural numbers,
and $\mathbf{R}$= the set of real numbers.

b.

Deduce that if I is a non-empty real interval, then C(I), the set of continuous functions $I \to \mathbf{R}$, has the cardinality of $\mathbf{R}$.
Remark. The proof shows more generally that if X is a non-empty separable Hausdorff space, then the cardinality of C(X) equals the cardinality of $\mathbf{R}$.

Solution by Dan Jurca

We recall the definitions: for sets A and B the set B^A equals the set of all functions from A to B, and a sequence in a set S is an element of $S^{\mathbf{N}}$; *i.e.*, a sequence in S is a function $\mathbf{N} \to S$. One way to verify the existence of a bijection $\mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ is to recall that there exists a bijection $\mathbf{R} \to 2^{\mathbf{N}}=\{0,1\}^{\mathbf{N}}$, where $2^{\mathbf{N}}=\{0,1\}^{\mathbf{N}}$ is the set of binary sequences-sequences of 0s and 1s. (A nice argument is in chapter 2 of *Notes on Set Theory* by Yiannis Moschovakis.) Now a function $\varphi:B \to C$ induces $\varphi_*{}^A:B^A \to C^A$ by $f \to \varphi \circ f$, and it is trivial to verify that if $\varphi:B \to C$ is a bijection with inverse $\psi:C \to B$, then $\psi_*{}^A:C^A \to B^A$ is the inverse of $\varphi_*{}^A$, so that $\varphi_*{}^A$ is also a bijection. Thus it suffices to show that there exists a bijection $(\{0,1\}^{\mathbf{N}})^{\mathbf{N}} \to \{0,1\}^{\mathbf{N}}$; *i.e.*, there exists a bijection from the set of sequences of binary sequences to the set of binary sequences. We can do this using Cantor's ``first diagonal method".

So suppose $X \in (\{0,1\}^{\mathbf{N}})^{\mathbf{N}}$; say $X=(\mathbf{x}_i)_{i=0}^{\infty}$, where each $\mathbf{x}_i$ is a binary sequence, say $\mathbf{x}_i=(x_{i,j})_{j=0}^{\infty}$, where each $x_{i,j} \in \{0,1\}$. We consider the doubly infinite array as follows.

$$
\begin{array}{ll}
\mathbf{x}_0 & =(x_{0,0} \quad x_{0,1} \quad x_{0,2} \quad x_{0,3} \quad \ldots) \\
\mathbf{x}_1 & =(x_{1,0} \quad x_{1,1} \quad x_{1,2} \quad x_{1,3} \quad \ldots) \\
\mathbf{x}_2 & =(x_{2,0} \quad x_{2,1} \quad x_{2,2} \quad x_{2,3} \quad \ldots) \\
\mathbf{x}_3 & =(x_{3,0} \quad x_{3,1} \quad x_{3,2} \quad x_{3,3} \quad \ldots) \\
\vdots & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots
\end{array}
$$

We associate with this sequence X of binary sequences the binary sequence $\mathbf{y}$, where

$$\mathbf{y}=(x_{0,0}\quad x_{0,1}\quad x_{1,0}\quad x_{0,2}\quad x_{1,1}\quad x_{2,0}\quad x_{0,3}\quad x_{1,2}\quad x_{2,1}\quad x_{3,0}\quad \ldots).$$

It is easy to see that **y** is well-defined, and that X can be recovered from **y**; *i.e.*, $(\{0,1\}^{\mathbf{N}})^{\mathbf{N}} \to \{0,1\}^{\mathbf{N}}$ by $X \to \mathbf{y}$ is a bijection. Thus there exists a bijection $\mathbf{R}^{\mathbf{N}} \to \mathbf{R}$.

Also solved by Bill Nico