Problem for 2006 June

Proposed by Dan Jurca

Prove that of all right triangles with given perimeter the isosceles right triangle has maximum area.

One can easily do this using techniques from calculus; can you find a proof which does not use calculus?

Solution by the proposer

We find the perimeter p and area A of a right triangle with base x and height y.

$$p = x + y + \sqrt{x^2+y^2}$$

$$A = xy/2, \quad \text{so that}$$

$$y = \frac{p^2-2px}{2p-2x} \quad \text{and}$$

$$A = \frac{p^2x-2px^2}{4p-4x}$$

Iff x=y, then $x=p/(2+\sqrt{2})$ and $A=x^2/2=p^2/(12+8\sqrt{2})$. Thus we show that

$$x < p \Rightarrow \frac{p^2x-2px^2}{4p-4x} \le \frac{p^2}{12+8\sqrt{2}}.$$

Now

$$0 \le (p-(2+\sqrt{2})x)^2, \text{ (equality iff } p=(2+\sqrt{2})x), \quad \text{so}$$

$$0 \le p^2-(4+2\sqrt{2})px+(6+4\sqrt{2})x^2, \quad \text{so}$$

$$
\begin{aligned}
(4+2\sqrt{2})px-(6+4\sqrt{2})x^2 &\le p^2, \quad \text{so}\\
4px+2\sqrt{2}px-(6+4\sqrt{2})x^2 &\le p^2, \quad \text{so}\\
3px+2\sqrt{2}px-6x^2-4\sqrt{2}x^2 &\le p^2-px, \quad \text{so}\\
(3+2\sqrt{2})(px-2x^2) &\le p^2-px, \quad \text{so}\\
(12+8\sqrt{2})(px-2x^2) &\le 4p^2-4px, \quad \text{so}\\
(12+8\sqrt{2})(p^2x-2px^2) &\le (4p-4x)p^2, \quad \text{and (since } x < p)\\
\frac{p^2x-2px^2}{4p-4x} &\le \frac{p^2}{12+8\sqrt{2}},
\end{aligned}
$$

as desired; clearly equality holds if and only if $x=p/(2+\sqrt{2})$.