Problem for 2006 July

Communicated by Dan Jurca

The following appears as problem **C** on page 76 of *General Topology* by John L. Kelley.

> A topological space is a *door space* iff every subset is either open or closed. A Hausdorff door space has at most one accumulation point, and if x is a point which is not an accumulation point, then {x} is open. (If U is an arbitrary neighbrhood of an accumulation point y, then U ~ {y} is open.)

Prove that each Hausdorff door space contains at most one accumulation point.

Solution by Dan Jurca

Suppose that X is a Hausdorff door space and $a \in X$ is an accumulation point. We shall show that if $x \in X$ and $x \neq a$, then $\{x\}$ is open, so that x is not an accumulation point.

Since X is Hausdorff and $a \neq x$ there exist open subsets U and V of X such that $a \in U$, $x \in V$, and $U \cap V = \emptyset$. Let $A = \{a\} \cup (V - \{x\})$. If A is open, then $A \cap U = \{a\}$ is open. This is impossible, since if $\{a\}$ is open then the open set $\{a\}$ contains a but no other point of X, contradicting the hypothesis that a is an accumulation point. Since X is a door space and A is not open, it follows that A is closed, so that $X - A$ is open; hence $(X-A) \cap V$ is also open. Now clearly $x \in (X-A) \cap V$, since $x \notin A$ but $x \in V$; *i.e.*, $\{x\} \subset (X-A) \cap V$. But if $y \in (X-A) \cap V$, then $y \in V$; also $y \in X-A$, so that $y \notin V - \{x\}$. Hence $y \in V - (V - \{x\}) = V \cap \{x\} = \{x\}$. That is, $(X-A) \cap V \subset \{x\}$. Hence the open set $(X-A) \cap V = \{x\}$, so that $\{x\}$ is open, and x is not an accumulation point.

Also solved by Massoud Malek