Problem for 2006 August

Proposed by Dan Jurca

Prove the following.

$$\lim_{n\to\infty} \int_0^{\pi/2} \sin^n\theta \, d\theta = 0$$

Solution by the proposer

Suppose $0 < \varepsilon < \pi/2$. Let $a=(\pi-\varepsilon)/2$, so $0 < a < \pi/2$. Now

$$\int_0^{\pi/2} \sin^n\theta \, d\theta = \int_0^{a} \sin^n\theta \, d\theta + \int_a^{\pi/2} \sin^n\theta \, d\theta.$$

Since $0 < \sin a < 1$, $\sin^n a \to 0$ as $n\to\infty$; hence there exists n_0 such that $1 \le n_0$ and $n_0 \le n \Rightarrow \sin^n a \le \varepsilon/(2a)$, so $n_0 \le n \Rightarrow a\sin^n a \le \varepsilon/2$. Therefore, since $1 \le n \Rightarrow \sin^n$ increases on the interval $[0,a]$,

$$n_0 \le n \Rightarrow \int_0^{a} \sin^n\theta \, d\theta < (a-0)\sin^n a$$

$$\le \varepsilon/2.$$

Also

$$\int_a^{\pi/2} \sin^n\theta \, d\theta < (\pi/2-a)\sin^n(\pi/2)$$

$$=(\pi/2-a)\times 1$$

$$=\varepsilon/2.$$

Therefore

$$n_0 \leq n \Rightarrow \int_0^{\pi/2} \sin^n\theta \, d\theta < \varepsilon, \quad \text{whence}$$

$$\lim_{n\to\infty} \int_0^{\pi/2} \sin^n\theta \, d\theta = 0.$$

Also solved by Massoud Malek and John M. Sayer