Problem for 2006 Novemeber

Proposed by Istvan Simon

One sunny day Mary did the following experiment on the Hayward campus of the California State University, East Bay: she marked the end of the shadow of a vertical flagpole on a piece of paper placed on the horizontal ground every five minutes for four hours. She was astonished to discover that the forty-eight points that she had marked on the paper were all on a straight line. Explain what Mary observed.

Solution by the proposer

1. The Sun is very far from the Earth. As a result of this, any two of the Sun's rays that hit the Earth simultaneously can be considered to be parallel for all practical purposes.
2. Imagine you are standing on the North Pole at the height of summer, on June 21. Because you are on axis with the Earth's rotation the rays of the Sun during the day will form a cone. The vertex of this cone is the tip of your head, and the end of your shadow will describe a circle. Now imagine you repeat the same experiment on the Hayward Campus, that is you stand in the parking lot for a while on June 21. What will you see?
3. Because the Sun's rays are always parallel everywhere, the rays of the Sun will form a cone parallel to the cone at the North Pole mentioned above. The difference is that the cone will be cut by the horizontal plane which is now at an angle with the cone, no longer perpendicular to the cone's axis, and the exact angle depends on the latitude where you are standing. It follows that the end of your shadow describes a conic section, that is the intersection of a cone and a plane. Depending on your latitude, the end of your shadow will describe a circle (at the North Pole), an ellipse at high latitudes, or an arc of a hyperbola, as you move further South from the North Pole. On the Hayward Campus it will be an arc of a hyperbola on June 21.
4. It follows from the above discussion that what Mary observed does not occur on every day. In particular, we just proved that on June 21, the shadow would not describe a straight line, as she observed, but a hyperbola.
5. To understand what Mary observed it is helpful to return to the North Pole and repeat the experiment on June 22, June 23, June 24, *etc.* … every day of the year. As the Earth moves around the Sun the angle of the cone will change, because the axis of the Earth is inclined 23.5 degrees off the perpendicular to the ecliptic, the plane of the orbit of the Earth. What you would see as a result is that the cone would open up as the Sun would move down closer to the horizon at the North Pole. Your shadow would be still on a

circle every day, but the circle would be opening up and becoming larger and larger with each passing day. During the equinox, the Sun is perpendicular at the equator. Therefore, at that point the cone degenerates into a plane perpendicular to the axis of the Earth. The intersection of this plane with the horizontal plane in Hayward is a straight line, and that is what Mary observed. Note that during the equinox (which occurs twice in a year), the shadow describes a straight line everywhere on Earth, as proved in this section.

6. To be absolutely precise, the shadow would describe a spiral on the North Pole, but we can ignore the small motion of the Earth around the Sun in its orbit on any one day as a very good approximation, as done in the discussion above. There is also the wobbling effect of the axis of the Earth, with a period of 22,000 years but this minute effect can be safely ignored on any one day of observations.

7. An interesting corollary of our proof is that we can infer that Mary'e experiment must have occurred on September 22 or March 22 approximately, the dates for the equinox in most years. (Note that the date of the equinox changes very slowly due to the wobbling of the axis of the Earth mentioned above in (6).)

Also partially solved by Oksana Master
