Problem for 2006 December

Proposed by Dan Jurca

The following is a slight variation on Mathematical Mayhem problem 213, which appeared (with solution) in the 2006 November edition of the Canadian mathematics journal *Crux Mathematicorum*.

Suppose x and y are (real or complex) numbers, and P equals the following product of n factors.

$$P=(x+y)(x^2+y^2)(x^4+y^4)(x^8+y^8)(x^{16}+y^{16})\dots$$

Find the value of P.

Solution by the proposer

First suppose x=y; then

$$\begin{aligned} P &= 2x\times 2x^2\times 2x^4\times\dots\times 2x^{2n-1} \\ &= 2^n x^{1+2+4+\dots+2n-1} \\ &= 2^n x^{2n-1}. \end{aligned}$$

Next

$$\begin{aligned} (x-y)P &= (x-y)(x+y)(x^2+y^2)(x^4+y^4)(x^8+y^8)\dots(x^{2n-1}+y^{2n-1}) \\ &= (x^2-y^2)(x^2+y^2)(x^4+y^4)(x^8+y^8)\dots(x^{2n-1}+y^{2n-1}) \\ &= (x^4-y^4)(x^4+y^4)(x^8+y^8)\dots(x^{2n-1}+y^{2n-1}) \\ &= (x^8-y^8)(x^8+y^8)\dots(x^{2n-1}+y^{2n-1}) \\ &\vdots \\ &= x^{2n}-y^{2n}. \end{aligned}$$

Hence

$$P=\begin{cases} 2^n x^{2n-1} & \text{if } x=y, \\ \dfrac{x^{2n}-y^{2n}}{x-y} & \text{if } x \neq y. \end{cases}$$

Remark. One observes that if $f(y)=y^{2n}-x^{2n}$, then

$$\lim_{y\to x} \frac{x^{2n}-y^{2n}}{x-y} = \lim_{y\to x} \frac{y^{2n}-x^{2n}}{y-x}$$
$$=f'(x)$$
$$=2^n x^{2n-1}.$$

Also solved by Grant Morgan