Problem for 2007 February

Proposed by Dan Jurca

For each $n,\ 1 \le n$, let

$$q_n = \frac{2 \times 5 \times 8 \times 11 \times 14 \times \cdots \times (3n-1)}{1 \times 4 \times 7 \times 10 \times 13 \times \cdots \times (3n-2)}.$$

Determine whether

$$\lim_{n\to\infty} q_n$$

exists, and, if the limit exists, its value.

Solution by the proposer

The limit does not exist (or equals ∞), as shown here. We have

$$\begin{aligned}
\ln q_n &= \ln \frac{2 \times 5 \times 8 \times \cdots \times (3n-1)}{1 \times 4 \times 7 \times \cdots \times (3n-2)} \\
&= \ln \frac{2}{1} + \ln \frac{5}{4} + \cdots + \ln \frac{3n-1}{3n-2} \\
&= \sum_{i=1}^{n} \ln \frac{3i-1}{3i-2} \\
&= \sum_{i=1}^{n} \ln \left(1 + \frac{1}{3i-2} \right).
\end{aligned}$$

From the well-known (and easily proved) inequality

$$-1 < x \ \Rightarrow\ \frac{x}{1+x} < \ln(1+x)$$

we have

$$\begin{aligned}
1 \le i \ &\Rightarrow\ \frac{\frac{1}{3i-2}}{1 + \frac{1}{3i-2}} < \ln \left(1 + \frac{1}{3i-2} \right), \quad \text{so that} \\
1 \le i \ &\Rightarrow\ \frac{1}{3i-1} < \ln \left(1 + \frac{1}{3i-2} \right), \quad \text{whence} \\
1 \le i \ &\Rightarrow\ \frac{1}{3i} < \ln \left(1 + \frac{1}{3i-2} \right).
\end{aligned}$$

Therefore

$$1 \le n \ \Rightarrow\ \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) = \frac{1}{3} H_n < \ln q_n,$$

where $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$, and since $H_n \to \infty$ as $n \to \infty$, it follows that $q_n \to \infty$ as well.

Also solved by Kelly Hubble, Massoud Malek, and Nathan Speed