Problem for 2007 March

Proposed by Dan Jurca

One can define the ``Möbius" function, μ:P→**C**, from the set P of positive integers to **C**, the set of complex numbers, as follows.

$$n \in P \Rightarrow \mu(n)=\begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } a^2|n \text{ for some positive integer } a \\ (-1)^k & \text{if } n=p_1p_2\ldots p_k\text{, where the } p_i \text{ are distinct primes} \end{cases}$$

A basic, important, and easily proved result in elementary number theory asserts that

$$\sum_{d|n} \mu(d)=\begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } 1 < n. \end{cases}$$

Show that this characterizes the function μ; *i.e.*, prove the following proposition.

If f:P→**C** and

$$\sum_{d|n} f(d)=\begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } 1 < n, \end{cases}$$

then f=μ.

Solution by the proposer

Letting n=1, we find

$$\sum_{d|1} f(d)=f(1)=1,$$

so that f(1)=μ(1). Now suppose that $1 < n$ and $1 \le d < n \Rightarrow f(d)=\mu(d)$. Then since, obviously, n|n, we have

$$\sum_{d|n} f(d) = 0, \quad \text{so that, solving for } f(n),$$

$$f(n) = -\sum_{d < n \text{ and } d|n} f(d)$$

$$= -\sum_{d < n \text{ and } d|n} \mu(d) \quad \text{by the inductive hypothesis}$$

$$= \mu(n), \quad \text{since } \sum_{d|n} \mu(d)=0,$$

whence f(n)=μ(n). Therefore, by (the strong form of) mathematical induction we have $1 \le n \Rightarrow f(n)=\mu(n)$, and thus f=μ.

Also solved by Grant Morgan and Jean Simutis