Problem for 2007 May

Communicated by Dan Jurca

The following problem involves a well-known fact, but may make an interesting exercise.

Show that the sequence $(x_n)_{n=1}^{\infty}$ where

$$x_n=\left(1+\frac{1}{n}\right)^n$$

strictly increases.

Solution by Dan Jurca

We shall show somewhat more generally that if $0 < a$ and $f:(0,\infty)\to\mathbf{R}$ by

$$f(x)=\left(1+\frac{a}{x}\right)^x$$

then f strictly increases. For $0 < f$ and

$$\ln f(x) = x\ln\left(1+\frac{a}{x}\right)$$

$$= x\ln\left(\frac{x+a}{x}\right), \quad \text{so}$$

$$\frac{1}{f(x)}f'(x) = \ln\left(\frac{x+a}{x}\right) + x\cdot\frac{x}{x+a}\cdot\frac{-a}{x^2}$$

$$= \ln\left(\frac{x+a}{}\right) - \frac{a}{}$$

$$\left(\quad x \quad \right) \qquad x+a$$

With φ:(0,∞)→**R** by

$$\varphi(x) = \ln\left(\frac{x+a}{x}\right) - \frac{a}{x+a}, \qquad \text{we have}$$

$$\begin{aligned}\varphi'(x) &= \frac{x}{x+a}\cdot\frac{-a}{x^2} + \frac{a}{(x+a)^2} \\ &= \frac{-a^2}{x(x+a)^2} \\ &< 0,\end{aligned}$$

so that φ strictly decreases. Since

$$\lim_{0^+} \varphi = \infty \quad \text{and} \quad \lim_{\infty} \varphi = 0$$

it follows that $0 < \varphi$, and since $0 < f$ it follows that $0 < f'$ as well. Thus f strictly increases, as asserted.

Also solved by Massoud Malek and John M. Sayer