Problem for 2007 June

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The following problem appears on page 235 of *The IMO Compendium*.

> Let n be a positive integer. Show that
>
> $$(\sqrt{2}+1)^n=\sqrt{m}+\sqrt{m-1}$$
>
> for some positive integer m.

Prove the following generalization.

If a and b are positive integers, then for each positive integer n there exists a positive integer m such that

$$(\sqrt{a}+\sqrt{b})^n=\sqrt{m}+\sqrt{m-(a-b)^n}\,.$$

Solution by the proposer

We shall solve for m and from the resulting expression will deduce that m is indeed a positive integer. First, write n=2q+r, where q is a nonnegative integer and r=0 or r=1. Next from

$$(\sqrt{a}+\sqrt{b})^n=\sqrt{m}+\sqrt{m-(a-b)^n} \quad \text{we have}$$

$$(\sqrt{a}+\sqrt{b})^n-\sqrt{m}=\sqrt{m-(a-b)^n}\,, \quad \text{so by squaring,}$$

$$(\sqrt{a}+\sqrt{b})^{2n}-2(\sqrt{a}+\sqrt{b})^n\sqrt{m}+m = m-(a-b)^n, \quad \text{so}$$

$$\sqrt{m} = \frac{(\sqrt{a}+\sqrt{b})^{2n}+(a-b)^n}{2(\sqrt{a}+\sqrt{b})^n}$$

$$= \frac{(\sqrt{a}+\sqrt{b})^n}{2} + \frac{1}{2}\left(\frac{a-b}{\sqrt{a}+\sqrt{b}}\right)^n$$

$$= \frac{(\sqrt{a}+\sqrt{b})^n}{2} + \frac{1}{2}\left(\frac{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}}\right)^n$$

$$= \frac{(\sqrt{a}+\sqrt{b})^n}{2} + \frac{(\sqrt{a}-\sqrt{b})^n}{2}$$

$$= \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\left(a^{(n-i)/2}b^{i/2}+(-1)^i a^{(n-i)/2}b^{i/2}\right)$$

$$= \sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{2i}a^{(n-2i)/2}b^i$$

$$= \sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{2i}a^{n/2-i}b^i$$

$$= \sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{2i}a^{(2q+r)/2-i}b^i$$

$$= a^{r/2}\sum_{i=0}^{q}\binom{n}{2i}a^{q-i}b^i, \quad \text{whence}$$

$$m = a^r\left(\sum_{i=0}^{q}\binom{n}{2i}a^{q-i}b^i\right)^2, \quad \text{a positive integer.}$$