Problem for 2007 September

Proposed by Dan Jurca

Dawson's function F is defined by the following integral.

$$F(x)=e^{-x^2}\int_0^x e^{t^2}dt$$

Determine whether

$$\int_0^\infty F \text{ converges; } \textit{i.e.}\text{, whether } \lim_{x\to\infty}\int_0^x F \text{ exists.}$$

Solution by the proposer

The integral does not converge. We have by l'Hôpital's rule and the fundamental theorem of calculus

$$\begin{aligned}
\lim_{x\to\infty} xF(x) &= \lim_{x\to\infty} \frac{x}{e^{x^2}}\int_0^x e^{t^2}dt \\
&= \lim_{x\to\infty} \frac{x\int_0^x e^{t^2}dt}{e^{x^2}} \\
&= \lim_{x\to\infty} \frac{\int_0^x e^{t^2}dt+xe^{x^2}}{2xe^{x^2}} \\
&= \lim_{x\to\infty} \frac{e^{x^2}+e^{x^2}+2x^2e^{x^2}}{2e^{x^2}+4x^2e^{x^2}}
\end{aligned}$$

$$= \lim_{x\to\infty} \frac{2+2x^2}{2+4x^2} \quad \text{so that}$$

$$\lim_{x\to\infty} xF(x) = 1/2.$$

Therefore there exists x_0 such that $0 < x_0$ and $x_0 \le x \Rightarrow 1/4 \le xF(x)$.
It follows that $x_0 \le x \Rightarrow 1/(4x) \le F(x)$. Hence $x_0 \le x \Rightarrow$

$$\int_{x_0}^{x} \frac{1}{4t}\,dt \le \int_{x_0}^{x} F \quad \text{so that}$$

$$\frac{1}{4}(\ln x - \ln x_0) \le \int_{x_0}^{x} F \quad \text{whence}$$

$$\lim_{x\to\infty} \int_0^x F = \infty.$$

Remark. We can now express this another way, as follows. We have

$$\lim_{x\to\infty} \frac{\int_0^x F}{\ln x} = \lim_{x\to\infty} \frac{F(x)}{1/x}$$

$$= \lim_{x\to\infty} xF(x)$$

$$= 1/2.$$

Therefore

$$\int^x F \sim \ 1 \ln x = \ln\sqrt{x}.$$

$$\int_{0} \qquad \frac{\quad}{2}$$

That is,

$$\lim_{x\to\infty} \frac{\int_{0}^{x} F}{\ln\sqrt{x}} = 1.$$

Also solved by Massoud Malek