Problem for 2007 October

Communicated by Dan Jurca

The following problem appears on page 76 of *The IMO Compendium* by Dušan Djukić, Vladimir Janković, Ivan Matić, and Nikola Petrović.

> Note that $8^3 - 7^3 = 169 = 13^2$ and $13 = 2^2 + 3^2$. Prove that if the difference between two consecutive cubes is a square, then it is the square of the sum of two consecutive squares.

Remark. Other examples follow.

n	$(n+1)^3 - n^3 = m^2$	$m = a^2 + (a+1)^2$	a
0	1	1	0
7	169	13	2
104	32761	181	9
1455	6355441	2521	35
20272	1232922769	35113	132
282359	239180661721	489061	494
3932760	46399815451081	6811741	1845
54776287	9001325016847969	94875313	6887
762935264	1746210653453054881	1321442641	25704

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Lemma. If x is a positive integer, then $12x^2 - 8x + 1$ is not the square of an integer.

Proof.

Since $12x^2 - 8x + 1 = (2x-1)(6x-1)$ and for each positive integer x $\gcd(2x-1, 6x-1) = 1$ (since $-(3x+1)(2x-1) + x(6x-1) = 1$), it follows that $12x^2 - 8x + 1$ is a square if and only if $2x-1$ is a square and $6x-1$ is a square. However, $6x - 1 = 6(x-1) + 5$ and no square is congruent to 5 mod 6, so $6x-1$ is not a square; therefore if x is a positive integer, then $12x^2 - 8x + 1$ is not a square, proving the lemma.

Now suppose n is a positive integer and $(n+1)^3 - n^3 = m^2$ for some positive integer m; then $3n^2+3n+1 = m^2$, so that $m^2 - 1 = 3n(n+1)$. Therefore $3|(m^2-1)$, and it follows that $3|(m-1)$ or $3|(m+1)$. We shall show that $3|(m-1)$ by showing that $3|(m+1)$ leads to a contradiction. For if $3|(m+1)$, then $m+1 = 3x$ for some positive integer x, so $m = 3x-1$. But from $m^2 = 3n^2 + 3n + 1$ we have $(3x-1)^2 = 3n^2 + 3n + 1$, whence $9x^2 - 6x = 3n^2 + 3n$, so $3x^2 - 2x = n^2 + n$, and $12x^2 - 8x = 4n^2 + 4n$, from which $12x^2 - 8x + 1 = 4n^2 + 4n + 1 = (2n+1)^2$, contradicting the lemma. Therefore 3 does not divide $m+1$, so that $3|(m-1)$.

Next, from $3n^2 + 3n + (1 - m^2) = 0$ we have (by the quadratic formula and since $0 < n$)

$$n = \frac{-3 + \sqrt{9 - 12(1-m^2)}}{6} = \frac{\sqrt{12m^2 - 3} - 3}{6} = \frac{\sqrt{3(4m^2-1)} - 3}{6} = \frac{\sqrt{3(2m-1)(2m+1)} - 3}{6},$$

so that $3(2m-1)(2m+1) = (6n+3)^2$, the square of an integer. Since $3|(m-1)$ there exists a positive integer q with $m = 3q+1$; hence $2m+1 = 6q+3$ and $3|(2m+1)$. Since $-(m+1)(2m-1) + m(2m+1) = 1$, it follows that $\gcd(2m-1, 2m+1) = 1$, and since $3|(2m+1)$ it follows that 3 does not divide $2m-1$. Therefore $2m-1$ is a square (for $(2m-1) \times 3(2m+1)$ is a square and is the product of two relatively prime integers); say $2m - 1 = s^2$. Then s is odd and

$$\left(\frac{s-1}{2}\right)^2 + \left(\frac{s-1}{2} + 1\right)^2 = \left(\frac{s-1}{2}\right)^2 + \left(\frac{s+1}{2}\right)^2 = \frac{(s-1)^2 + (s+1)^2}{4} = \frac{2s^2+2}{4} = \frac{s^2+1}{2} = m,$$

so that with $a = (s-1)/2$ we have $m = a^2 + (a+1)^2$, the sum of consecutive squares.