Problem for 2007 November

Proposed by Dan Jurca

Since there exists no elementary function F such that $F'(x)=e^{x^2}$, one computes approximations of integrals such as

$$\int_0^{10} e^{x^2}\,dx$$

using some numerical method. Similarly (or otherwise), find an accurate approximation (with at least four significant digits) of the following definite integral.

$$\int_0^{10} e^{2^x}\,dx$$

Solution by the proposer

Suppose $0 < a$, $1 \le ab$, and

$$I=\int_0^b e^{e^{ax}}\,dx.$$

(With a=ln2 and b=10 we recover the given problem.)

Proposition 1. $0 \le n \Rightarrow$

$$\int_0^b e^{e^{ax}}\,dx=\left[\frac{e^{e^{ax}}}{a}\sum_{i=1}^{n}\frac{(i-1)!}{e^{iax}}\right]_0^b+n!\int_0^b e^{-nax}e^{e^{ax}}\,dx$$

Proof.

The equation obviously holds if $n=0$, so suppose that $1 \le n$ and

$$I=\left[\frac{e^{eax}}{a}\sum_{i=1}^{n-1}\frac{(i-1)!}{e^{iax}}\right]_0^b+(n-1)!\int_0^b e^{-(n-1)ax}e^{eax}\,dx.$$

Let $u=e^{-nax}/a$ and $dv=e^{eax}\cdot ae^{ax}\,dx$; then $du=e^{-nax}\cdot -n\,dx$ and $v=e^{eax}$. Hence by parts

$$\begin{aligned} I &=\left[\frac{e^{eax}}{a}\sum_{i=1}^{n-1}\frac{(i-1)!}{e^{iax}}\right]_0^b+(n-1)!\left[\left.\frac{e^{eax}\cdot e^{-nax}}{a}\right|_0^b+n\int_0^b e^{-nax}e^{eax}\,dx\right] \\ &=\left[\frac{e^{eax}}{a}\sum_{i=1}^{n}\frac{(i-1)!}{e^{iax}}\right]_0^b+n!\int_0^b e^{-nax}e^{eax}\,dx, \end{aligned}$$

and the proposition follows by induction on n.

Now we let

$$A_n=\frac{e^{eab}}{a}\sum_{i=1}^{n}\frac{(i-1)!}{e^{iab}}\qquad s_n=\frac{e}{a}\sum_{i=1}^{n}(i-1)!\qquad \text{and}\qquad R_n=n!\int_0^b e^{-nax}e^{eax}\,dx.$$

Then $I=A_n-s_n+R_n$, and we show next that $I \approx A_n$ for some values of n. By this we mean that with

$$\text{rel err}_n=\frac{I-A_n}{I}$$

one can find a small n for which $|\text{rel err}_n| < 5\times10^{-4}$, for example.

Proposition 2.

$$1 \le n \text{ and } \frac{n+2}{n} e^{(n+1)ab+1} \le e^{eab} \Rightarrow 0 < \text{rel err}_n < \frac{n!ab}{e^{(n-1)ab}}$$

Proof.

First we observe that if the conditions on n above hold, then $e^{(n+1)ab+1} < e^{eab}$, so that $(n+1)ab+1 < e^{ab}$, whence $nab < e^{ab}$, so that (since $1 \le ab$) we have $n < e^{ab}$. It follows that $0 \le \ln n/a < b$.

Now let $\varphi:[0,b]\to\mathbf{R}$ by $\varphi(x)=e^{-nax}e^{eax}$. Then $\varphi(0)=e$, $\varphi(b)=e^{-nab}e^{eab}$, and

$$\begin{aligned} \varphi'(x) &= (-na+ae^{ax})\varphi(x) \\ &= a(e^{ax}-n)\varphi(x). \end{aligned}$$

Thus φ\searrow in [0,(lnn)/a], φ\nearrow in [(lnn)/a,b], so that $\varphi_{max}=\max\{\varphi(0),\varphi(b)\}=\max\{e,e^{-nab}e^{eab}\}=e^{-nab}e^{eab}$; for from $n < e^{ab}$ we have $n < (e^{ab}-1)/(ab)$, (since $1 \le ab$) so $nab < e^{ab}-1$ and $nab+1 < e^{ab}$, whence $e^{nab}\cdot e < e^{eab}$, so that $e < e^{-nab}e^{eab}$. It follows that $R_n < n!be^{-nab}e^{eab}$.

We next show that with n as in the hypothesis it follows that $0 \le R_n-s_n$. First by an easy induction on n it follows that $s_n \le (e/a)\cdot 2(n-1)!$. Next with $u=e^{-nax}e^{-ax}/a$, $dv=e^{eax}e^{ax}a\,dx$ and by parts again

$$\begin{aligned} \int_0^b e^{-nax}e^{eax}\,dx &= \left[\frac{e^{-(n+1)ax}}{a}\cdot e^{eax}\right]_0^b +(n+1)\int_0^b e^{-(n+1)ax}e^{eax}\,dx \\ &= \frac{e^{-(n+1)ab}\cdot e^{eab}}{a} - \frac{e}{a} +(n+1)\int_0^b e^{-(n+1)ax}e^{eax}\,dx; \quad \text{therefore} \end{aligned}$$

$$n!\left(\frac{e^{-(n+1)ab}\cdot e^{eab}}{a} - \frac{e}{a}\right) < R_n, \quad \text{and since } -(e/a)\cdot 2(n-1)! \le -s_n \quad \text{we have}$$

$$n!\left(\frac{e^{-(n+1)ab}\cdot e^{eab}}{a} - \frac{e}{a}\right) - \frac{e}{a}\cdot 2(n-1)! \le R_n-s_n, \quad \text{so}$$

$$\frac{(n-1)!}{a}\left(ne^{-(n+1)ab}\cdot e^{eab}-(n+2)e\right) \le R_n-s_n,$$

and since $(n+2)/n \cdot e^{(n+1)ab+1} \le e^{eab}$, it follows that $0 \le R_n - s_n$. Hence

$$\frac{I-A_n}{I} = \frac{(A_n-s_n+R_n)-A_n}{A_n-s_n+R_n}$$

$$= \frac{R_n-s_n}{A_n+(R_n-s_n)} \quad \text{so that}$$

$$0 \le \frac{I-A_n}{I} \le \frac{R_n}{A_n}. \quad \text{But since (obviously)}$$

$$\frac{e^{eab}}{a} \cdot \frac{1}{e^{ab}} \le A_n, \quad \text{we have}$$

$$0 \le \frac{I-A_n}{I} < \frac{n!be^{-nab}e^{eab}}{e^{eab}/a \cdot (1/e^{ab})}$$

$$= \frac{n!ab}{e^{(n-1)ab}},$$

proving proposition 2.

Now with a=ln2, b=10, and $1 \le n \le 146$ the conditions in the hypothesis of proposition 2 hold; in particular we let n=8. Then $I \approx A_8$ with a relative error less than 3×10^{-16}; hence with at least sixteen significant digits we have

$$\int_0^{10} e^{2x}\,dx \approx A_8$$

$$= \frac{e^{e\ln 2 \times 10}}{\ln 2} \sum_{i=1}^{8} \frac{(i-1)!}{e^{i \times \ln 2 \times 10}}$$

$$= \frac{e^{1,024}}{\ln 2} \times \left[\frac{1}{2^{10}} + \frac{1}{2^{20}} + \frac{2}{2^{30}} + \frac{6}{2^{40}} + \frac{24}{2^{50}} + \frac{120}{2^{60}} + \frac{720}{2^{70}} + \frac{5,040}{2^{80}} \right]$$

$$\approx 7.35950813976599166\ldots\times10^{441}.$$

Remark 1. A_{146} approximates the integral with at least 182 significant digits.

Remark 2. The function ψ: $[1,\infty)\to\mathbf{R}$ by $\psi(x)=(x+2)/x\cdot e^{(x+1)ab+1}$ strictly increases and (obviously) approaches ∞. Therefore there exists a maximum n satisfying the conditions in proposition 2.