Problem for 2008 January

Communicated by Dan Jurca

Prove that if

$a \in \mathbf{R},\ b \in \mathbf{R},\ c \in \mathbf{R}$;
$a < b < c$;
$a + b + c = 2$; and
$ab + bc + ca = 1$;

then

$0 < a < 1/3$;
$1/3 < b < 1$; and
$1 < c < 4/3$.

Solution by Dan Jurca

Since $c = 2-a-b$ we have from $ab+bc+ca = 1$ that $ab+b(2-a-b)+(2-a-b)a = 1$, so that $a^2+b^2+ab-2a-2b+1 = 0$, or $b^2+(a-2)b+(a^2-2a+1) = 0$. Therefore $b = (-a+2\pm\sqrt{a^2-4a+4-4a^2+8a-4})/2 = (-a+2\pm\sqrt{4a-3a^2})/2$. Since $b < c = 2-a-b$ we have $2b < 2-a$, so that if $b = -a+2+\sqrt{4a-3a^2})/2$, then $2b = -a+2+\sqrt{4a-3a^2} < 2-a$, so that $\sqrt{4a-3a^2} < 0$, which is impossible. Therefore $b = (-a+2-\sqrt{4a-3a^2})/2$. Now $3a^2 \le 4a$, so if $a < 0$ then $4 \le 3a$, whence $4/3 \le a$, which is contradictory; therefore $0 \le a$. But if $a = 0$, then $b = 1$, so $1 < c$, contradicting $a+b+c = 2$. Since $a < b$ we have $2a < -a+2-\sqrt{4a-3a^2}$, so $3a-2 < -\sqrt{4a-3a^2}$. Since the function $x \mapsto x^2$ decreases in the interval $(-\infty, 0]$ it follows that $4a-3a^2 < 9a^2-12a+4$, whence $0 < 12a^2-16a+4$ so $0 < 3a^2-4a+1 = (3a-1)(a-1)$. Hence either ($a < 1/3$ and $a < 1$) or ($1/3 < a$ and $1 < a$). Since $a \le 1$ (from $0 < a < b < c$ and $a+b+c = 2$) we must have $a < 1/3$. Thus $0 < a < 1/3$.

Now consider $\varphi : [0, 1/3] \to \mathbf{R}$ by $\varphi(x) = (-x+2-\sqrt{4x-3x^2})/2$. We find $\varphi(0) = 1,\ \varphi(1/3) = 1/3$, and

$$\begin{aligned}\varphi'(x) &= \frac{1}{2}\left(-1-\frac{2-3x}{\sqrt{4x-3x^2}}\right)\\ &= -\frac{1}{2}\frac{\sqrt{4x-3x^2}+2-3x}{\sqrt{4x-3x^2}}\\ &< 0\end{aligned}$$

so that φ decreases; hence, since $b = \varphi(a)$, it follows that $1/3 < b < 1$.

Next, $c = 2-a-b = 2-a-(2-a-\sqrt{4a-3a^2})/2 = (2-a+\sqrt{4a-3a^2})/2$. With $\psi : [0, 1/3] \to \mathbf{R}$ by $\psi(x) = (2-x+\sqrt{4x-3x^2})/2$ we find that $\psi(0) = 1,\ \psi(1/3) = 4/3$, and if $0 < x < 1/3$ then

$$\begin{aligned}\psi'(x) &= \frac{1}{2}\left(-1+\frac{2-3x}{\sqrt{4x-3x^2}}\right)\\ &= \frac{2-3x-\sqrt{4x-3x^2}}{2\sqrt{4x-3x^2}}.\end{aligned}$$

Now with $\theta : [0, 1/3] \to \mathbf{R}$ by $\theta(x) = 4x-3x^2$ we find $\theta(0) = 0,\ \theta(1/3) = 1$, and θ increases (since $0 < \theta'$), so that $\theta_{\max} = \theta(1/3) = 1/3$. From this it follows that $0 < \psi'$, so ψ increases in $[0, 1/3]$, and, finally, since $c = \psi(a),\ 1 < c < 4/3$.

Also solved by Massoud Malek and Grant Morgan