Problem for 2008 February

Communicated by Dan Jurca

The following problem appears on page 19 of *A Course in Enumeration* by Martin Aigner.

> **1.20** Let $a_n = \frac{1}{\binom{n}{0}} + \frac{1}{\binom{n}{1}} + \cdots + \frac{1}{\binom{n}{n}}$. Show that $a_n = \frac{n+1}{2n}a_{n-1} + 1$ and compute $\lim_{n\to\infty} a_n$ (if the limit exists). Hint: $a_n > 2 + \frac{2}{n}$ and $a_{n+1} < a_n$ for $n \geq 4$.

Solution by Dan Jurca

We recall that if $0 \leq n$, then

$$0 \leq m \leq n \;\Rightarrow\; \binom{n}{m} = \binom{n}{n-m} \quad \text{and} \quad 0 \leq m < n \;\Rightarrow\; \binom{n}{m} = \frac{n}{n-m}\binom{n-1}{m}.$$

Thus

$$\begin{aligned}
a_n &= \frac{1}{\binom{n}{0}} + \frac{1}{\binom{n}{1}} + \frac{1}{\binom{n}{2}} + \cdots + \frac{1}{\binom{n}{n-1}} + \frac{1}{\binom{n}{n}} \\
&= \frac{1}{\frac{n}{n-0}\binom{n-1}{0}} + \frac{1}{\frac{n}{n-1}\binom{n-1}{1}} + \frac{1}{\frac{n}{n-2}\binom{n-1}{2}} + \cdots + \frac{1}{\frac{n}{n-(n-1)}\binom{n-1}{n-1}} + 1 \\
&= \frac{1}{n}\left[\frac{n-0}{\binom{n-1}{0}} + \frac{n-1}{\binom{n-1}{1}} + \frac{n-2}{\binom{n-1}{2}} + \cdots + \frac{n-(n-1)}{\binom{n-1}{n-1}}\right] + 1 \\
&= a_{n-1} - \frac{1}{n}\left[\frac{0}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \frac{2}{\binom{n-1}{2}} + \cdots + \frac{n-1}{\binom{n-1}{n-1}}\right] + 1 \\
&= a_{n-1} - \frac{1}{n}\left[\frac{n-1}{\binom{n-1}{0}} + \frac{n-2}{\binom{n-1}{1}} + \frac{n-3}{\binom{n-1}{2}} + \cdots + \frac{0}{\binom{n-1}{n-1}}\right] + 1, \quad \text{so} \\
2a_n &= 2a_{n-1} - \frac{n-1}{n}a_{n-1} + 2, \quad \text{whence} \\
a_n &= a_{n-1} - \frac{n-1}{2n}a_{n-1} + 1 \\
&= \left(1 - \frac{n-1}{2n}\right)a_{n-1} + 1 \\
&= \frac{n+1}{2n}a_{n-1} + 1.
\end{aligned}$$

Now obviously

$$\begin{aligned}
4 \leq n \;\Rightarrow\; \frac{1}{\binom{n}{0}} + \frac{1}{\binom{n}{1}} + \frac{1}{\binom{n}{n-1}} + \frac{1}{\binom{n}{n}} &< a_n, \quad \text{so} \\
2 + \frac{2}{n} &< a_n.
\end{aligned}$$

Therefore

$$
\begin{aligned}
4 \le n \;\Rightarrow\;& 2 + \frac{2}{n} < a_n \\
\Rightarrow\;& 2n + 2 < n a_n \\
\Rightarrow\;& 1 < \frac{n}{2n+2} a_n \\
\Rightarrow\;& 1 < \frac{2n+2-n-2}{2(n+1)} a_n \\
\Rightarrow\;& 1 < \left[1 - \frac{n+2}{2(n+1)}\right] a_n \\
\Rightarrow\;& \frac{n+2}{2(n+1)} a_n + 1 < a_n \\
\Rightarrow\;& a_{n+1} < a_n.
\end{aligned}
$$

It follows that the sequence $(a_n)_{n=4}^{\infty}$, bounded below (by 2), decreases, hence converges. But if $(a_n) \to L$, then since

$$
\begin{aligned}
a_n &= \frac{n+1}{2n} a_{n-1} + 1, \quad \text{we have} \\
L &= \frac{1}{2} L + 1, \quad \text{whence} \\
L &= 2; \quad \text{so that, finally,} \\
\lim_{n\to\infty} a_n &= 2.
\end{aligned}
$$

Remarks. By induction on n one can show that

1. $0 \le n \;\Rightarrow\; a_n = \frac{n+1}{2^{n+1}} \sum_{i=1}^{n+1} \frac{2^i}{i}$, and
2. $8 \le n \;\Rightarrow\; 2 + \frac{2}{n} < a_n < 2 + \frac{3}{n}$.

Also solved by Massoud Malek