Problem for 2008 March

Communicated by Dan Jurca

Given 2n points in the plane with no three collinear, n colored blue and n colored red, prove that one can always connect the points in pairs by line segments having different colored endpoints and such that no two line segments cross each other.

Solution by Dan Jurca

The assertion is obviously true if n=1, hence follows by induction on n from the following

Proposition. If n is an integer, $2 \le n$, and there exist 2n points in the plane with no three collinear, n colored blue and n colored red, then there exists a (straight) line \l in the plane and an integer m, $1 \le m < n$, such that there are exactly m points colored blue and m points colored red on one side of \l, and n–m points colored blue and n–m points colored red on the other side of \l.

Proof.

The assertion is clearly true if n=2, so suppose $3 \le n$. Let S be the set of 2n points, let D be a disk which includes S, let T be the set of all n(2n–1) lines determined by each pair of points in S, regardless of color, and let P be a point in the plane not in D and not on any line in T. (Such a P clearly exists since T is a finite set of lines.) Since no three points in S are collinear a line determined by P and a point in S contains no other point of S. Therefore a line through P rotated through 180° meets the points in S one at a time. So suppose k is a line through P such that the disk D is on one side of k, and let c=0. Without loss of generality suppose that as k is rotated through 180° the first point in S which is on k is a point colored blue. Then as k rotates through 180° add 1 to c each time a point colored blue is met, and subtract 1 from c each time a point colored red is met. (Thus, c is a ``counter".) Clearly after k has rotated through 180° the value of c will be 0 again (since there are as many points colored blue as are colored red). If the last point met by k was colored blue, then c must have been −1 before k met this last point, hence must have been zero somewhere. Then l can be a line through P in the angle determined by k after c has assumed the value 0 for the first time (after being 1 earlier), and then either 1 again, or −1. A similar argument can be made if the first and last points met by k are colored red. If, however, the first and last points met by k are colored differently, then by considering the smaller set of points obtained by deleting these two points, and arguing as above, the proposition follows by induction on n.

Better solution communicated (but not found) by Dan Jurca

Suppose the points colored blue are labeled $B_1, B_2, \ldots, B_n$ and the points colored red are labeled $R_1, R_2, \ldots, R_n$, let S_n be the set of all permutations on the set $\{1, 2, \ldots, n\}$, and let $|PQ|$ be the length of the segment in the plane with endpoints P and Q. Let $\varphi: S_n \to \mathbf{R}$ by

$$\varphi(\pi) = \sum_{i=1}^{n} |B_i R_{\pi(i)}|.$$

Since S_n is a finite set the function φ attains a minimum value, say at $\pi_0 \in S_n$. Now if the segment $B_i R_{\pi0(i)}$ intersects the segment $B_j R_{\pi0(j)}$, then (by the triangle inequality) a value less than $\varphi(\pi_0)$ is attained by $\varphi(\pi_1)$, where $1 \le k \le n$, $k \ne i$, $k \ne j \Rightarrow \pi_1(k)=\pi_0(k)$, $\pi_1(i)=\pi_0(j)$, and $\pi_1(j)=\pi_0(i)$. This contradiction shows that no two of the segments $B_1 R_{\pi0(1)}$, $B_2 R_{\pi0(2)}$, …,\$B_n R_{\pi0(n)}$ intersect.

Also solved by Bojan Basic and Vlad Dumitriu
