Problem for 2008 April

Proposed by Dan Jurca

A method often used to solve an equation, or, equivalently, to find a zero of a function f:**R**→**R** (*i.e.*, a number z such that f(z)=0), is to compute the first few terms of the sequence $(x_n)_{n=0}^{\infty}$ where x_0 is chosen near z and for $1 \le n$ the terms x_n are generated using Newton's method. That is

$$1 \le n \Rightarrow x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

If f is nice enough and x_0 is sufficiently close to z the sequence so generated converges to z, and often converges (in an appropriate sense) rapidly.

1. Consider f:**R**→**R** by $f(x)=(4x^5-19x^4+26x^3-7x^2-4x+4)/4$, and observe that f(2)=0. Show that with $x_0=0$ the sequence of Newton's method iterates as described above is (0,1,0,1,0,1,…), hence does not converge.
2. Generalize this as follows. Suppose n is an integer and $2 \le n$. Then there exists a function f:**R**→**R** such that f(n)=0 but if we set $x_0=0$, then the sequence of Newton's method iterates as described above is

   $$(0,1,2,\dots,n-1,0,1,2,\dots,n-1,0,1,2,\dots,n-1,\dots,\dots);$$

   i.e., $x_i = i \bmod n$.

Solution by the proposer

1. We find

   $$1 \le i \Rightarrow x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$$

$$=x_{i-1}-\frac{4x_{i-1}^5-19x_{i-1}^4+26x_{i-1}^3-7x_{i-1}^2-4x_{i-1}+4}{20x_{i-1}^4-76x_{i-1}^3+78x_{i-1}^2-14x_{i-1}-4}$$

$$=\frac{16x_{i-1}^5-57x_{i-1}^4+52x_{i-1}^3-7x_{i-1}^2-4}{20x_{i-1}^4-76x_{i-1}^3+78x_{i-1}^2-14x_{i-1}-4},$$

so that if $x_{i-1}=0$, then $x_i=1$, and if $x_{i-1}=1$, then $x_i=0$.

2. There exists a unique polynomial p(x) of degree ≤ 2n+1 (a *Hermite interpolating polynomial*) with the following properties.

a. $p(0)=p(1)=p(2)=\ldots=p(n-2)=1$;

b. $p'(0)=p'(1)=p'(2)=\ldots=p'(n-2)=-1$;

c. $p(n-1)=1$, and $p'(n-1)=1/(n-1)$;

d. $p(n)=0$, and $p'(n)=0$.

This polynomial has the properties required in 2.

An example for n=3:

$$f(x)=\frac{-11x^7+192x^6-1142x^5+3066x^4-3851x^3+1962x^2-216x+216}{216}$$

Also solved by Bojan Basic (Novi Sad, Serbia), Jan van Delden, Minhhua Lin (Xi'an, China), and Massoud Malek

Bojan Basic found an expression for a function f as required in 2 with coefficients satisfying a certain linear system, and proved that there exists a solution of the system.

Minhua Lin gave the following explicit formula for a function f satisfying the conditions in 2.

$$f(x)=\begin{cases} e^{-x} & \text{if } x\in(-\infty,n-2) \\ (x+3-n)(x+1-n)^2e^{-(n-2)}+ \\ (3x-2nx+2n^2-4n+2)(x+2-n)^2(n-1) & \text{if } x\in[n-2,n-1) \\ (2xn-x-2n^2+4n-2)(x-n)^2(n-1) & \text{if } x\in[n-1,\infty) \end{cases}$$