Problem for 2008 May

Communicated by Dan Jurca

According to the *Hungarian Problem Book II* the following problem appeared in the 1913 Eötvös mathematics competition.

Prove that for every integer $n > 2$

$$(1\cdot 2\cdot 3\ldots n)^2 > n^n.$$

Solution by Dan Jurca

$1 < i < n \Rightarrow$

$$\begin{aligned} i &< n \quad \text{so} \\ (i-1)i &< (i-1)n \quad \text{so} \\ i^2-i &< in-n \quad \text{so} \\ n &< in-i^2+i \\ &= i(n-i+1) \quad \text{so} \\ \frac{n}{n+1-i} &< i. \end{aligned}$$

Therefore $2 < n \Rightarrow$

$$\frac{n^n}{n!} = \prod_{i=1}^{n} \frac{n}{i}$$

$$= \quad n-1 \quad \frac{n}{\ }$$

$$\prod_{i=1} i$$

$$= \prod_{i=2}^{n} \frac{n}{n+1-i}$$

$$< \prod_{i=2}^{n} i$$

$$= \prod_{i=1}^{n} i$$

$$= n!, \quad \text{whence}$$

$$n^n < (n!)^2,$$

as desired.

Also solved by Alexandru Cardaniuc, Jinzhong Li (China), Minghua Lin (China), Grant Morgan, John Sayer, and Armend Shabhani (Republic of Kosova)
