Problem for 2008 June

Proposed by Dan Jurca

Recall that the vector ``cross product" $\mathbf{a}\times\mathbf{b}$ in $\mathbf{R}^3$ is not associative. For example

$$\begin{aligned} \mathbf{i}\times(\mathbf{i}\times\mathbf{j}) &= \mathbf{i}\times\mathbf{k} = -\mathbf{j}, \quad \text{but} \\ (\mathbf{i}\times\mathbf{i})\times\mathbf{j} &= \mathbf{0}\times\mathbf{j} = \mathbf{0}. \end{aligned}$$

So for vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^3$ let

$$\begin{aligned} V_1(\mathbf{a},\mathbf{b}) &= \{\mathbf{x}\in \mathbf{R}^3 \mid \mathbf{x}\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{x}\times\mathbf{a})\times\mathbf{b}\} \\ V_2(\mathbf{a},\mathbf{b}) &= \{\mathbf{x}\in \mathbf{R}^3 \mid \mathbf{a}\times(\mathbf{x}\times\mathbf{b}) = (\mathbf{a}\times\mathbf{x})\times\mathbf{b}\} \\ V_3(\mathbf{a},\mathbf{b}) &= \{\mathbf{x}\in \mathbf{R}^3 \mid \mathbf{a}\times(\mathbf{b}\times\mathbf{x}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{x}\}. \end{aligned}$$

1. Prove that $\mathbf{a},\mathbf{b} \in \mathbf{R}^3 \Rightarrow V_i(\mathbf{a},\mathbf{b})$ is a subspace of $\mathbf{R}^3$, i=1,2,3.
2. Prove that $\mathbf{a},\mathbf{b} \in \mathbf{R}^3 \Rightarrow V_1(\mathbf{a},\mathbf{b}) = V_3(\mathbf{b},\mathbf{a})$.
3. Prove that $\mathbf{a} \in \mathbf{R}^3 \Rightarrow V_2(\mathbf{a},\mathbf{a}) = \mathbf{R}^3$.
4. What are the subspaces $V_i(\mathbf{a},\mathbf{0})$, i=1,2,3?
5. If $\mathbf{a} \neq \mathbf{0}$, what is the subspace $V_3(\mathbf{a},\mathbf{a})$?
6. What is the subspace $V_3(\mathbf{i},\mathbf{j})$?
7. What is the subspace $V_3(\mathbf{j},\mathbf{i})$?
8. What is the subspace $V_3(\mathbf{i}+\mathbf{j},\mathbf{i})$?
9. Show that $\mathbf{a},\mathbf{b} \in \mathbf{R}^3 \Rightarrow V_i(\mathbf{a},\mathbf{b}) \neq \{\mathbf{0}\}$, i=1,2,3.

Solution by the proposer

1.

Obviously $\mathbf{0}$, the zero vector, is in each $V_i(\mathbf{a},\mathbf{b})$, so each is a nonempty subset of $\mathbf{R}^3$. If $\mathbf{x} \in V_1(\mathbf{a},\mathbf{b})$ and $\mathbf{y} \in V_1(\mathbf{a},\mathbf{b})$, then

$$\begin{aligned}(\mathbf{x}+\mathbf{y})\times(\mathbf{a}\times\mathbf{b}) &= (\mathbf{x}\times(\mathbf{a}\times\mathbf{b}))+(\mathbf{y}\times(\mathbf{a}\times\mathbf{b})) \\ &= ((\mathbf{x}\times\mathbf{a})\times\mathbf{b})+((\mathbf{y}\times\mathbf{a})\times\mathbf{b}) \\ &= ((\mathbf{x}\times\mathbf{a})+(\mathbf{y}\times\mathbf{a}))\times\mathbf{b} \\ &= ((\mathbf{x}+\mathbf{y})\times\mathbf{a})\times\mathbf{b},\end{aligned}$$

so that $\mathbf{x}+\mathbf{y} \in V_1(\mathbf{a},\mathbf{b})$. If $k \in \mathbf{R}$ and $\mathbf{x} \in V_1(\mathbf{a},\mathbf{b})$, then

$$\begin{aligned}(k\mathbf{x})\times(\mathbf{a}\times\mathbf{b}) &= k\times(\mathbf{x}\times(\mathbf{a}\times\mathbf{b})) \\ &= k((\mathbf{x}\times\mathbf{a})\times\mathbf{b}) \\ &= (k(\mathbf{x}\times\mathbf{a}))\times\mathbf{b} \\ &= ((k\mathbf{x})\times\mathbf{a})\times\mathbf{b},\end{aligned}$$

so that $k\mathbf{x} \in V_1(\mathbf{a},\mathbf{b})$. Hence $V_1(\mathbf{a},\mathbf{b})$ is a subspace of $\mathbf{R}^3$; and similarly $V_2(\mathbf{a},\mathbf{b})$ and $V_3(\mathbf{a},\mathbf{b})$ are subspaces of $\mathbf{R}^3$.

2.

If $\mathbf{x} \in V_1(\mathbf{a},\mathbf{b})$, then $\mathbf{x}\times(\mathbf{a}\times\mathbf{b})=(\mathbf{x}\times\mathbf{a})\times\mathbf{b}$; hence

$$\begin{aligned}\mathbf{b}\times(\mathbf{a}\times\mathbf{x}) &= -\mathbf{b}\times(\mathbf{x}\times\mathbf{a}) \\ &= (\mathbf{x}\times\mathbf{a})\times\mathbf{b} \\ &= \mathbf{x}\times(\mathbf{a}\times\mathbf{b}) \\ &= -(\mathbf{a}\times\mathbf{b})\times\mathbf{x} \\ &= (\mathbf{b}\times\mathbf{a})\times\mathbf{x},\end{aligned}$$

so that $x \in V_3(\mathbf{b},\mathbf{a})$, and $V_1(\mathbf{a},\mathbf{b}) \subset V_3(\mathbf{b},\mathbf{a})$. Similarly $V_3(\mathbf{b},\mathbf{a}) \subset V_1(\mathbf{a},\mathbf{b})$, whence $V_1(\mathbf{a},\mathbf{b})=V_3(\mathbf{b},\mathbf{a})$.

3.

$\mathbf{x} \in \mathbf{R}^3 \Rightarrow \mathbf{a}\times(\mathbf{x}\times\mathbf{a})=-(\mathbf{x}\times\mathbf{a})\times\mathbf{a}=(\mathbf{a}\times\mathbf{x})\times\mathbf{a}$, so that $\mathbf{x} \in V_2(\mathbf{a},\mathbf{a})$; hence $\mathbf{R}^3 \subset V_2(\mathbf{a},\mathbf{a})$, so $V_2(\mathbf{a},\mathbf{a})=\mathbf{R}^3$.

4.

Since $\mathbf{x} \in \mathbf{R}^3 \Rightarrow \mathbf{x}\times(\mathbf{a}\times\mathbf{0})=\mathbf{x}\times\mathbf{0}=\mathbf{0}=(\mathbf{x}\times\mathbf{a})\times\mathbf{0}$, $\mathbf{x} \in V_1(\mathbf{a},\mathbf{0})$, and $V_1(\mathbf{a},\mathbf{0})=\mathbf{R}^3$. Similarly $V_2(\mathbf{a},\mathbf{0})=\mathbf{R}^3$ and $V_3(\mathbf{a},\mathbf{0})=\mathbf{R}^3$.

5.

$\mathbf{x} \in V_3(\mathbf{a},\mathbf{a}) \Rightarrow \mathbf{a}\times(\mathbf{a}\times\mathbf{x})=(\mathbf{a}\times\mathbf{a})\times\mathbf{x}=\mathbf{0}\times\mathbf{x}=\mathbf{0}$; it follows that $(\mathbf{a}\cdot\mathbf{x})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{x}=\mathbf{0}$, so that (since $\mathbf{a} \neq \mathbf{0}$) $\mathbf{x}=(\mathbf{a}\cdot\mathbf{x})\mathbf{a}/(\mathbf{a}\cdot\mathbf{a})$; *i.e.*, $\mathbf{x}=k\mathbf{a}$ for some $k \in \mathbf{R}$. Conversely, if $\mathbf{x}=k\mathbf{a}$, then $\mathbf{a}\times(\mathbf{a}\times\mathbf{x})=\mathbf{a}\times(\mathbf{a}\times k\mathbf{a})=\mathbf{0}=\mathbf{0}\times\mathbf{x}=(\mathbf{a}\times\mathbf{a})\times\mathbf{x}$, and $\mathbf{x} \in V_3(\mathbf{a},\mathbf{a})$. Hence $\mathbf{a} \neq \mathbf{0} \Rightarrow V_3(\mathbf{a},\mathbf{a})=\mathbf{R}\mathbf{a}$.

6.

Suppose $\mathbf{x}=x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}$ and $\mathbf{x}\in V_3(\mathbf{i},\mathbf{j})$. Then $\mathbf{i}\times(\mathbf{j}\times(x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}))=(\mathbf{i}\times\mathbf{j})\times(x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k})$. Therefore $\mathbf{i}\times(-x_1\mathbf{k}+x_3\mathbf{i})=\mathbf{k}\times(x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k})$, so $x_1\mathbf{j}=x_1\mathbf{j}-x_2\mathbf{i}$. Therefore $x_2=0$ and $\mathbf{x}=x_1\mathbf{i}+x_3\mathbf{k}$. Conversely, $\mathbf{i}\times(\mathbf{j}\times(x_1\mathbf{i}+x_3\mathbf{k}))=\mathbf{i}\times(-x_1\mathbf{k}+x_3\mathbf{i})=x_1\mathbf{j}=x_1(\mathbf{k}\times\mathbf{i})+\mathbf{0}=\mathbf{k}\times(x_1\mathbf{i}+x_3\mathbf{k})=(\mathbf{i}\times\mathbf{j})\times(x_1\mathbf{i}+x_3\mathbf{k})$, so $x_1\mathbf{i}+x_3\mathbf{k}\in V_3(\mathbf{i},\mathbf{j})$. Therefore $V_3(\mathbf{i},\mathbf{j})=\mathbf{Ri}\oplus\mathbf{Rk}$.

7.

If $\mathbf{x}=x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}\in V_3(\mathbf{j},\mathbf{i})$, then $\mathbf{j}\times(\mathbf{i}\times(x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}))=\mathbf{j}\times(x_2\mathbf{k}-x_3\mathbf{j})=x_2\mathbf{i}=(\mathbf{j}\times\mathbf{i})\times(x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k})=-\mathbf{k}\times(x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k})=-x_1\mathbf{j}+x_2\mathbf{i}$, so that $x_1=0$ and $\mathbf{x}=x_2\mathbf{j}+x_3\mathbf{k}$. Conversely, if $\mathbf{x}=x_2\mathbf{j}+x_3\mathbf{k}$, then $\mathbf{j}\times(\mathbf{i}\times\mathbf{x})=\mathbf{j}\times(\mathbf{i}\times(x_2\mathbf{j}+x_3\mathbf{k}))=\mathbf{j}\times(x_2\mathbf{k}-x_3\mathbf{j})=x_2\mathbf{i}=-\mathbf{k}\times(x_2\mathbf{j}+x_3\mathbf{k})=(\mathbf{j}\times\mathbf{i})\times(x_2\mathbf{j}+x_3\mathbf{k})$, so $\mathbf{x}\in V_3(\mathbf{j},\mathbf{i})$. Hence $V_3(\mathbf{j},\mathbf{i})=\mathbf{Rj}\oplus\mathbf{Rk}$.

8.

If $\mathbf{x}=x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}$ and $\mathbf{x}\in V_3(\mathbf{i}+\mathbf{j},\mathbf{i})$, then we find $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}\times(x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}))=(\mathbf{i}+\mathbf{j})\times(x_2\mathbf{k}-x_3\mathbf{j})=-x_2\mathbf{j}-x_3\mathbf{k}+x_2\mathbf{i}=x_2\mathbf{i}-x_2\mathbf{j}-x_3\mathbf{k}$; and $((\mathbf{i}+\mathbf{j})\times\mathbf{i})\times(x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k})=-\mathbf{k}\times(x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k})=-x_1\mathbf{j}+x_2\mathbf{i}=x_2\mathbf{i}-x_1\mathbf{j}$. Therefore $x_1=x_2$ and $x_3=0$, so that $\mathbf{x}=k(\mathbf{i}+\mathbf{j})$ for some $k\in\mathbf{R}$. Conversely, $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}\times k(\mathbf{i}+\mathbf{j}))=k(\mathbf{i}+\mathbf{j})\times\mathbf{k}=k(-\mathbf{j}+\mathbf{i})=-k(\mathbf{j}-\mathbf{i})=-k\mathbf{k}\times(\mathbf{i}+\mathbf{j})=((\mathbf{i}+\mathbf{j})\times\mathbf{i})\times k(\mathbf{i}+\mathbf{j})$, so $k(\mathbf{i}+\mathbf{j})\in V_3(\mathbf{i}+\mathbf{j},\mathbf{i})$. Thus $V_3(\mathbf{i}+\mathbf{j},\mathbf{i})=\mathbf{R}(\mathbf{i}+\mathbf{j})$.

9.

Suppose $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$. Then by a straightforward computation we find that $\mathbf{x}\in V_1(\mathbf{a},\mathbf{b})$ if and only if $M_1\mathbf{x}=\mathbf{0}$, $\mathbf{x}\in V_2(\mathbf{a},\mathbf{b})$ if and only if $M_2\mathbf{x}=\mathbf{0}$, and $\mathbf{x}\in V_3(\mathbf{a},\mathbf{b})$ if and only if $M_3\mathbf{x}=\mathbf{0}$ where M_1, M_2, and M_3 are the following matrices.

$$M_1=\begin{bmatrix} a_2b_2+a_3b_3 & -a_2b_1 & -a_3b_1 \\ -a_1b_2 & a_1b_1+a_3b_3 & -a_3b_2 \\ -a_1b_3 & -a_2b_3 & a_1b_1+a_2b_2 \end{bmatrix}$$

$$M_2=\begin{bmatrix} 0 & a_1b_2-a_2b_1 & a_1b_3-a_3b_1 \\ -a_1b_2+a_2b_1 & 0 & a_2b_3-a_3b_2 \\ -a_1b_3+a_3b_1 & -a_2b_3+a_3b_2 & 0 \end{bmatrix}$$

$$M_3=\begin{bmatrix} a_2b_2+a_3b_3 & -a_1b_2 & -a_1b_3 \end{bmatrix}$$

$$
\begin{bmatrix}
-a_2b_1 & a_1b_1+a_3b_3 & -a_2b_3 \\
-a_3b_1 & -a_3b_2 & a_1b_1+a_2b_2
\end{bmatrix}
$$

By another straightforward computation one finds that each of M_1, M_2, and M_3 is a singular matrix; hence for each **a** and **b** there exists a nonzero vector in each of $V_1(\mathbf{a},\mathbf{b})$, $V_2(\mathbf{a},\mathbf{b})$, and $V_3(\mathbf{a},\mathbf{b})$.

Also solved by Minghua Lin (China)
