Problem for 2008 August

Proposed by Dan Jurca

Do there exist sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ of positive integers such that

i. x_i+y_i equals the square of an integer, and

ii. $i \neq j \Rightarrow x_i+y_j$ does not equal the square of an integer?

Solution by the proposer

Yes, as shown by the following example. For i=1,2,3,… let $x_i=y_i=2^{2i-1}$. Thus $(x_i)_{i=1}^{\infty}=(y_i)_{i=1}^{\infty}=(2^1, 2^3, 2^7, 2^{15}, \ldots)=(2, 8, 128, 32768, \ldots)$. Then $1 \leq i \Rightarrow$

$$\begin{aligned} x_i+y_i &= 2^{2i-1}+2^{2i-1} \\ &= 2\times 2^{2i-1} \\ &= 2^{2i} \\ &= 2^{2i-1\times 2} \\ &= (2^{2i-1})^2, \end{aligned}$$

the square of an integer. However, if $i \neq j$, say $1 \leq i < j$, then

$$\begin{aligned} x_i+y_j &= 2^{2i-1}+2^{2j-1} \\ &= 2^{2i-1}(1+2^{2j-2i}), \end{aligned}$$

and since 2^i-1 is an odd number $(2\times 2^{i-1}-1)$, the first factor consists of an odd number of factors of the prime 2, and the second factor is an odd number $(2\times 2^{2j-2i-1}+1)$, the product does not equal the square of an integer.

A similar result holds for any integer a, $1 \le a$, and $1 \le i \Rightarrow x_i=(2a)^{2i}/2$, or any rearrangement of any subsequence of the sequence $(x_i)_{i=1}^{\infty}$; and there exist many more cases where $(x_i)=(y_i)$ and x_i+y_j equals the square of an integer if and only if i=j.

Also solved by Bojan Basic (Serbia), Mahmoud Ezzaki (Morocco), Minghua Lin (China), Massoud Malek, and Grant Morgan

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