Problem for 2008 November

Communicated by Dan Jurca

Prove: $1 \le n \Rightarrow \sum_{k=1}^{n}[((-1)^{k-1})/k](n \parallel k)=\sum_{k=1}^{n}{}^{1}/_{k}$.

Solution by Dan Jurca

Consider the polynomial function f_n:$\mathbf{R}\rightarrow\mathbf{R}$ by

$$
\begin{aligned}
f_n(x) &= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} x^{k-1} \\
&= \frac{1}{-x} \sum_{k=1}^{n} \binom{n}{k} (-x)^{k} \quad \text{if } x \neq 0 \\
&= \frac{1}{-x} \left(-1+ \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot (-x)^{k} \right) \\
&= \frac{1}{-x} (-1+(1-x)^{n}). \quad \text{Then}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} &= \int_0^1 f_n \\
&= \int_0^1 \frac{1-(1-x)^{n}}{x}\, dx.
\end{aligned}
$$

With H_n equal to this integral we next show

$$H_n = {}^{n} \quad \frac{1}{} .$$

$$\sum_{k=1} k$$

This clearly holds if n=1; if $2 \le n$ and

$$H_{n-1} = \sum_{k=1}^{n-1} \frac{1}{k}, \quad \text{then}$$

$$H_n = \int_0^1 \frac{1-(1-x)(1-x)^{n-1}}{x} dx$$

$$= \int_0^1 \frac{1-(1-x)^{n-1}}{x} dx + \int_0^1 (1-x)^{n-1} dx$$

$$= H_{n-1} + \frac{1}{n},$$

so the result follows by induction on n.

Solution by Bojan Basi'c

We prove

$$x \in \mathbf{R} \Rightarrow -\sum_{k=1}^{n} \frac{x^k}{k} \binom{n}{k} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{(x+1)^k}{k}.$$

To show these polynomial functions are equal it is sufficient to prove that they are equal at a point and that their first derivatives are equal. They are obviously equal at x=0 (each side evaluates to 0); next

$$\Big(- \sum^{n} \frac{x^k}{k} \binom{n}{k} \Big)' = - \sum^{n} x^{k-1} \binom{n}{k}$$

$$\Big(\sum_{k=1} \frac{\quad}{k} \binom{\ }{k} \Big) \qquad \sum_{k=1} \binom{\ }{k}$$

$$= -\frac{\sum_{k=0}^{n} x^k \binom{n}{k} - 1}{x}$$

$$= -\frac{(x+1)^n - 1}{x}, \quad \text{and}$$

$$\left(\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{(x+1)^k}{k} \right)' = -\sum_{k=1}^{n} (x+1)^{k-1}$$

$$= -\frac{(x+1)^n - 1}{(x+1) - 1}$$

$$= -\frac{(x+1)^n - 1}{x},$$

so the derivatives are equal. The assertion of the problem is the special case $x=-1$.

Also solved by Bojan Basi'c (Serbia), Massoud Malek, and John Sayer

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