Problem for 2008 December

Proposed by Dan Jurca

Show that the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is odd if and only if n is 1 less than a power of 2.

Solution 1 by the proposer

We shall use the following notation. For each positive integer n and each prime number p we let $\nu_p(n)$ equal the number of occurrences of p in the factorization of n as a product of primes; *i.e.*,

$$n = 2^{\nu_2(n)} \times 3^{\nu_3(n)} \times 5^{\nu_5(n)} \times \ldots.$$

It is obvious that for each prime p we have $\nu_p(mn)=\nu_p(m)+\nu_p(n)$; and we recall that for each positive integer n and each prime p

$$\nu_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots.$$

First we consider the case of positive and even n. By a straightforward induction we find that

$$\binom{2n}{n} = \frac{2^{n/2}}{(n/2)!} \times ((n+1)\times(n+3)\times(n+5)\times\ldots\times(2n-1)) \quad \text{so that}$$

$$\nu_2\left(\binom{2n}{n}\right) = n/2 - \nu_2((n/2)!)$$

$$= \frac{n}{2} - \left(\left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \left\lfloor \frac{n}{16} \right\rfloor + \ldots \right) .$$

Now we observe that the sum in the parentheses, being a finite sum, is strictly less than the infinite sum n/4+n/8+n/16+… = n/2; it follows that

$$0 < \nu_2\left(\binom{2n}{n} \right) ,$$

and since n+1 is odd, it follows that if n is positive and even, then $0 < \nu_2(C_n)$, so that C_n is even.

Next suppose n is positive and odd. By a similar induction we find

$$\binom{2n}{n} = \frac{2^{(n+1)/2}}{((n-1)/2)!} \times((n+2)\times(n+4)\times(n+6)\times\ldots\times(2n-1)) \quad \text{so that}$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{2^{(n+1)/2}}{(n+1)((n-1)/2)!} \times((n+2)\times(n+4)\times(n+6)\times\ldots\times(2n-1))$$

$$= \frac{2^{(n-1)/2}}{((n+1)/2)((n-1)/2)!} \times((n+2)\times(n+4)\times(n+6)\times\ldots\times(2n-1))$$

$$= \frac{2^{(n-1)/2}}{((n+1)/2)!} \times((n+2)\times(n+4)\times(n+6)\times\ldots\times(2n-1)) \quad \text{whence}$$

$$\nu_2(C_n) = (n-1)/2 - \nu_2(((n+1)/2)!)$$

$$= \frac{n-1}{2} - \left(\left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n+1}{8} \right\rfloor + \left\lfloor \frac{n+1}{16} \right\rfloor + \ldots \right) .$$

We have $C_{2^0-1}=C_0=1=C_1=C_{2^1-1}$ so suppose that $n=2^i-1$ for some i, $2 \le i$. Then the sum in the parentheses equals $2^{i-2}+2^{i-3}+\ldots+1=2^{i-1}-1=(n-1)/2$, so that $\nu_2(C_n)=0$, and C_n is odd.

Otherwise, if n is odd but not of the form 2^i-1, then there exist (unique) positive integers p and q such that $n=2^p+q$ and $0 < q < 2^p-1$. Hence the sum in the parentheses equals

$$(2^{p-2}+\lfloor (q+1)/4\rfloor)+(2^{p-3}+\lfloor (q+1)/8\rfloor)+\ldots+(1+\lfloor (q+1)/2^p\rfloor)$$

$$=2^{p-1}-1+(\lfloor (q+1)/4\rfloor+\lfloor (q+1)/8\rfloor+\ldots)$$

$$< 2^{p-1}-1+(q+1)/2$$

$$=(2^p+q)/2-1/2$$

$$=n/2-1/2$$

$$=(n-1)/2$$

so that $0 < \nu_2(C_n)$, and C_n is even.

Solution 2 by the proposer

It is well-known that C_n is the number of binary trees with n nodes. Now for each binary tree T let T′ be the binary tree obtained from T by interchanging the left and right subtrees at each node; restricting this to binary trees with n nodes we have a bijection ϕ_n from the set of binary trees with n nodes to itself, and we observe that $\phi_n \circ \phi_n$ is the identity. Further we observe that $\phi_n(T)=T$ if and only if T is a full binary tree with n nodes, and this is possible if and only if $n=2^i-1$ for some natural number i. It follows at once that the number of binary trees with n nodes, C_n, is odd if and only if n is 1 less than a power of 2.

Also solved by Bojan Basic (Serbia) and John M. Sayer