Problem for 2009 January

Proposed by Dan Jurca

Suppose A is the following $n \times n$ matrix, where $A = (a_{ij})$, and $a_{ij} = 1 + \delta_{ij}$.

$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 2 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 2 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 2 \end{pmatrix}$$

a. Find the determinant of A.
b. Find a basis of $\mathbf{R}^n$ consisting of eigenvectors of A.

Solution by the proposer

For $1 \le i \le n$ let $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbf{R}^n$. Here the '1' appears in the i-th component. Also let $\mathbf{x}_1 = \mathbf{e}_1 + \cdots + \mathbf{e}_n$, the vector in $\mathbf{R}^n$ with '1' in each component, and for $2 \le i \le n$ let $\mathbf{x}_i = \mathbf{e}_i - \mathbf{e}_1$. A straightforward computation shows that $A\mathbf{e}_i = \mathbf{e}_i + \mathbf{x}_1$. Therefore $A\mathbf{x}_1 = A(\mathbf{e}_1 + \cdots + \mathbf{e}_n) = \mathbf{e}_1 + \cdots + \mathbf{e}_n + n\mathbf{x}_1 = (n+1)\mathbf{x}_1$, and $2 \le i \le n \;\Rightarrow\; A\mathbf{x}_i = A(\mathbf{e}_i - \mathbf{e}_1) = A\mathbf{e}_i - A\mathbf{e}_1 = (\mathbf{e}_i + \mathbf{x}_1) - (\mathbf{e}_1 + \mathbf{x}_1) = \mathbf{e}_i - \mathbf{e}_1 = \mathbf{x}_i$. It follows that $1 \le i \le n \;\Rightarrow\; \mathbf{x}_i$ is an eigenvector of A.

Suppose $k_1 \in \mathbf{R}, \ldots, k_n \in \mathbf{R}$ and $k_1\mathbf{x}_1 + \cdots + k_n\mathbf{x}_n = \mathbf{0}$. Then (by definition of $\mathbf{x}_i$) we have $(k_1 - k_2 - \cdots - k_n)\mathbf{e}_1 + (k_1 + k_2)\mathbf{e}_2 + \cdots + (k_1 + k_n)\mathbf{e}_n = \mathbf{0}$. Since $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a linearly independent set (in fact, the "standard basis" of $\mathbf{R}^n$) it follows that

$$\begin{aligned} k_1 - k_2 - \cdots - k_n &= 0, \\ k_1 + k_2 &= 0, \\ &\vdots \\ k_1 + k_n &= 0. \end{aligned}$$

Hence $2 \le i \le n \;\Rightarrow\; k_i = -k_1$; then from the first equation $nk_1 = 0$. Therefore $k_1 = 0$ and also $2 \le i \le n \;\Rightarrow\; k_i = 0$. Thus $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a linearly independent set, and hence is a basis of $\mathbf{R}^n$.

Therefore the characteristic polynomial of A is $p(\lambda) = (\lambda - (n+1))(\lambda - 1)^{n-1}$, and the determinant of $A = n + 1$.

Also solved by Bojan Bašić (Serbia) and Massoud Malek