Problem for 2009 February

Proposed by Dan Jurca

For $1 \le n$ let $\theta_n = \dfrac{\pi}{2n+1}$, and prove $\displaystyle\sum_{k=1}^{n} \sin^2(k\theta_n) = \frac{2n+1}{4}$.

Solution by the proposer

Since

$$\begin{aligned}
\sin k\alpha &= \frac{e^{ik\alpha} - e^{-ik\alpha}}{2i} \quad \text{(where } i^2 = -1\text{) we have} \\
\sin^2 k\alpha &= -\frac{(e^{2i\alpha})^k - 2 + (e^{-2i\alpha})^k}{4}, \quad \text{so} \\
\sum_{k=1}^{n} \sin^2(k\alpha) &= \frac{n}{2} - \frac{1}{4}\left(\sum_{k=1}^{n}(e^{2i\alpha})^k + \sum_{k=1}^{n}(e^{-2i\alpha})^k\right) \\
&= \frac{n}{2} - \frac{1}{4}\left(\frac{e^{2(n+1)i\alpha} - e^{2i\alpha}}{e^{2i\alpha} - 1} + \frac{e^{-2(n+1)i\alpha} - e^{-2i\alpha}}{e^{-2i\alpha} - 1}\right) \\
&= \frac{n}{2} - \frac{1}{4}\frac{(e^{-2i\alpha} - 1)(e^{2(n+1)i\alpha} - e^{2i\alpha}) + (e^{2i\alpha} - 1)(e^{-2(n+1)i\alpha} - e^{-2i\alpha})}{(e^{2i\alpha} - 1)(e^{-2i\alpha} - 1)} \\
&= \frac{n}{2} - \frac{1}{4}\frac{(e^{2ni\alpha} + e^{-2ni\alpha}) - (e^{2(n+1)i\alpha} + e^{-2(n+1)i\alpha}) + (e^{2i\alpha} + e^{-2i\alpha}) - 2}{2 - (e^{2i\alpha} + e^{-2i\alpha})} \\
&= \frac{n}{2} - \frac{1}{4}\frac{2\cos 2n\alpha - 2\cos 2(n+1)\alpha + 2\cos 2\alpha - 2}{2 - 2\cos 2\alpha} \\
&= \frac{n}{2} - \frac{1}{4}\frac{\cos 2n\alpha - \cos 2(n+1)\alpha + \cos 2\alpha - 1}{1 - \cos 2\alpha} \\
&= \frac{n}{2} - \frac{1}{4}\left(\frac{\cos 2(n+1)\alpha - \cos 2n\alpha}{\cos 2\alpha - 1} - 1\right) \\
&= \frac{n}{2} - \frac{1}{4}\left(\frac{-2\sin((2n+1)\alpha)\sin\alpha}{\cos 2\alpha - 1} - 1\right), \quad \text{so with } \alpha = \theta_n = \frac{\pi}{2n+1} \\
\sum_{k=1}^{n} \sin^2(k\theta_n) &= \frac{n}{2} + \frac{1}{4}\left(\frac{2\sin\pi\ \sin(\pi/(2n+1))}{\cos(2\pi/(2n+1)) - 1} + 1\right) \\
&= \frac{n}{2} + \frac{1}{4} \quad \text{(since } \sin\pi = 0\text{)} \\
&= \frac{2n+1}{4},
\end{aligned}$$

as desired.

Also solved by Bojan Bašić (Serbia), Jan van Delden (the Netherlands), and John M. Sayer