Problem for 2009 March

Proposed by Dan Jurca

Find positive integers $a,\ b,\ c$, and d such that $a^4 + b^5 + c^6 = d^7$.

Solution by the proposer

If $a = 3^{75},\ b = 3^{60},\ c = 3^{50}$, and $d = 3^{43}$, then

$$\begin{aligned} a^4 + b^5 + c^6 &= (3^{75})^4 + (3^{60})^5 + (3^{50})^6 \\ &= 3^{300} + 3^{300} + 3^{300} \\ &= 3 \times 3^{300} \\ &= 3^{301} \\ &= 3^{43\times 7} \\ &= (3^{43})^7 \\ &= d^7. \end{aligned}$$

More generally consider positive integers $e_1,\ e_2,\ \ldots,\ e_n$, and f where f divides $1 + k \times \operatorname{lcm}\{e_1, e_2, \ldots, e_n\}$ for some positive integer k. Then there exist infinitely many solutions $(x_1,\ x_2,\ \ldots,\ x_n,\ y)$ in positive integers of

$$x_1^{e_1} + x_2^{e_2} + \cdots + x_n^{e_n} = y^f.$$

For if $e = \operatorname{lcm}\{e_1, e_2, \ldots, e_n\}$ and $1 + ke = qf$, then for each nonnegative integer r, if $x_i = n^{(k+rf)e/e_i}$ (a positive integer since e_i divides e) and $y = n^{q+re}$, then

$$\begin{aligned} x_1^{e_1} + x_2^{e_2} + \cdots + x_n^{e_n} &= (n^{(k+rf)e/e_1})^{e_1} + (n^{(k+rf)e/e_2})^{e_2} + \cdots + (n^{(k+rf)e/e_n})^{e_n} \\ &= n^{(k+rf)e} + n^{(k+rf)e} + \cdots + n^{(k+rf)e} \\ &= n \times n^{(k+rf)e} \\ &= n^{1+(k+rf)e} \\ &= n^{(1+ke)+rfe} \\ &= n^{qf+ref} \\ &= n^{(q+re)f} \\ &= (n^{q+re})^f \\ &= y^f. \end{aligned}$$

The special case $n = 3,\ e_1 = 4,\ e_2 = 5,\ e_3 = 6$, and $f = 7$ yields the stated problem. $(e = 60,\ k = 5,\ 1 + ke = 301 = 43 \times 7,\ q = 43$, and we choose $r = 0$.)

Also solved by Bojan Bašić (Serbia)