Problem for 2009 May

Proposed by Dan Jurca

1. Show that there exist exactly two matrices A such that

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

2. Show that if $2 \le n$ and I_n equals the $n \times n$ identity matrix, then there exist uncountably many matrices A such that $A^2 = I_n$.

Solution by the proposer

1. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

then $a^2 + bc = 1,\ ab + bd = 1,\ ac + cd = 0$, and $bc + d^2 = 1$. From the first and fourth of these we have $a^2 - d^2 = 0$, so that $(a - d)(a + d) = 0$, and $a = -d$ or $a = d$. If $a = -d$, then the second equation becomes $0 = 1$, so that $a = d$. Hence the second equation yields $2ab = 1$ and the third becomes $2ac = 0$. Since $a \ne 0$, it follows that $c = 0$. Therefore $a^2 = 1$, so that $a = \pm 1$, and $b = \pm 1/2$. Finally, $d = a = \pm 1$. Hence

$$A = \begin{bmatrix} -1 & -1/2 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}.$$

2. If $x \in \mathbf{R}$ (or $x \in \mathbf{C}$), then

$$\begin{bmatrix} x & x^2 - 1 \\ -1 & -x \end{bmatrix} \times \begin{bmatrix} x & x^2 - 1 \\ -1 & -x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so there exist uncountably many matrices A such that $A^2 = I_2$. Now suppose that $2 < n$ and B is a matrix such that $B^2 = I_{n-1}$. Then (where $\mathbf{0}$ is the column matrix)

$$\begin{bmatrix} B & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \times \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} I_{n-1} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix};$$

hence if

$$A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix},$$

then $A^2 = I_n$ and the assertion follows by induction on n.

Also solved by Bojan Bašić (Serbia), Michael Mortensen, and Murray Stokely