Problem for 2009 July

Proposed by Dan Jurca

Does there exist an infinite-dimensional normed linear space $(X, \| \ \|)$ such that each subspace of X is closed?

Solution by the proposer

No, as shown in the following

Proposition. If $(X, \| \ \|)$ is an infinite dimensional normed space, then X includes a subspace which is not closed.

Proof.

Suppose $\mathcal{B}$ is a (Hamel) basis for X. Since by hypothesis $\mathcal{B}$ is infinite there exists a denumerable subset $\{\mathbf{b}_0, \mathbf{b}_1, \ldots\}$ of $\mathcal{B}$ and since each $\mathbf{b}_j \neq \mathbf{0}$, we can let $\mathbf{x}_0 = \mathbf{b}_0$ and $1 \leq j \ \Rightarrow \ \mathbf{x}_j = 1/(j(j+1)) \cdot \mathbf{b}_j/\|\mathbf{b}_j\|$. Hence $1 \leq j \ \Rightarrow \ \|\mathbf{x}_j\| = 1/(j(j+1))$.

For $0 \leq n$ let $Y_n = \{k_0\mathbf{x}_0 + k_1\mathbf{x}_1 + \cdots + k_n\mathbf{x}_n \in X \mid k_0 + k_1 + \cdots + k_n = 0\}$. Then each Y_n is an n-dimensional subspace of X, and $Y_0 \leq Y_1 \leq \cdots \leq Y_n$. Let $Y = \bigcup\{Y_n \mid 0 \leq n\}$. Then Y is a subspace of X, and Y is not closed, as follows.

For $1 \leq n$ let $\mathbf{y}_n = \mathbf{x}_0 - (1/n)(\mathbf{x}_1 + \cdots + \mathbf{x}_n)$. Then $\mathbf{y}_n \in Y_n \leq Y$, and $(\mathbf{y}_n)_{n=1}^{\infty} \to \mathbf{x}_0$, since

$$\begin{aligned} 1 \leq n \ \Rightarrow \ \|\mathbf{x}_0 - \mathbf{y}_n\| &= \|(1/n)(\mathbf{x}_1 + \cdots + \mathbf{x}_n)\| \\ &= (1/n)\|\mathbf{x}_1 + \cdots + \mathbf{x}_n\| \\ &\leq (1/n)(\|\mathbf{x}_1\| + \cdots + \|\mathbf{x}_n\|) \\ &= \frac{1}{n}\sum_{j=1}^{n} \frac{1}{j(j+1)} \\ &= \frac{1}{n}\left(1 - \frac{1}{n+1}\right) \\ &< \frac{1}{n}. \end{aligned}$$

However, $\forall n: \ \mathbf{x}_0 \notin Y_n$, so $\mathbf{x}_0 \notin Y$; hence Y is not a closed subspace of X.

Remark. The factors $1/(j(j+1))$ above can of course be replaced with any a_j where $\sum_{j=1}^{\infty} a_j$ is a convergent series of positive terms.

Solution by Bojan Bašić (Serbia)

The answer is no, that is: *Every infinite-dimensional normed linear space* $(X, \| \cdot \|)$ *has a subspace which is not closed.* In order to prove this, we are going to prove that the kernel of a linear functional f on a normed space always form a subspace, and even more, that if f is unbounded then the corresponding subspace is not closed, thereby providing the answer by giving a construction of an unbounded f on a given space.

It is almost trivial to see that $\ker f$ form a subspace. Indeed, for any $x, y \in \ker f$ and $a, b \in \mathbf{R}$ it holds $f(ax + by) = af(x) + bf(y) = 0 + 0 = 0$, and therefore $ax + by \in \ker f$.

Assume now that f is unbounded. Choose a sequence $x_n \in X$ such that $(\forall n \in \mathbf{N})(\|x_n\| \leq 1)$ and $\lim_{n\to\infty} f(x_n) = \infty$ (what is possible by unboundedness of f). Further, choose x which is not in $\ker f$, and notice that for each $n \in \mathbf{N}$ it holds $f\left(x - \frac{f(x)}{f(x_n)}x_n\right) = f(x) - \frac{f(x)}{f(x_n)}f(x_n) = 0$, therefore $x - \frac{f(x)}{f(x_n)}x_n \in \ker f$. However, since it is easy to see that $\lim_{n\to\infty}\left(x - \frac{f(x)}{f(x_n)}x_n\right) = x$, we have just found a sequence from $\ker f$ which converges to an element not in $\ker f$, what means that $\ker f$ is not closed.

Finally, let us show how to construct an unbounded f. Choose a sequence y_n of linearly independent elements of X, and let $f(y_n) = n\|y_n\|$. By the axiom of choice, sequence y_n can be extended to a vector space basis of X, and we could define f in the other elements of this basis to be equal to 0. This definition on the basis uniquely determines f on each $y \in X$, and this f is obviously unbounded.