Problem for 2009 August

Proposed by Dan Jurca

$$\int \frac{dx}{x^9 - x}$$

Solution by the proposer

We will consider more generally

$$\int \frac{dx}{x^\alpha - x}$$

where $\alpha \neq 1$. Let $u = 1/x$, so $x = 1/u$ and $dx = -1/u^2\, du$. Then

$$\begin{aligned}
\int \frac{dx}{x^\alpha - x} &= \int \frac{-\dfrac{1}{u^2}\, du}{\dfrac{1}{u^\alpha} - \dfrac{1}{u}} \\
&= \int \frac{u^{\alpha-2}\, du}{u^{\alpha-1} - 1} \\
&= \frac{1}{\alpha - 1} \int \frac{d(u^{\alpha-1} - 1)}{u^{\alpha-1} - 1} \\
&= \frac{1}{\alpha - 1} \ln |u^{\alpha-1} - 1| + C \\
&= \frac{1}{\alpha - 1} \ln \left| \frac{1}{x^{\alpha-1}} - 1 \right| + C \\
&= \ln |x^{1-\alpha} - 1| / (\alpha - 1) + C.
\end{aligned}$$

Also solved by Hans de Moor, Jinzhong Li (China), John Sayer, and Jan van Delden (the Netherlands)