## Problem for 2009 September

Communicated by Dan Jurca

An observer at the point $(a, b, c) \neq (0,0,0)$ in $\mathbf{R}^3$ looks toward the origin; what angles will the observer see between the positive coordinate axes?

For example, an observer at the point $(1,1,1)$ looking toward the origin will see angles of $2\pi/3 = 120^\circ$ between the axes.

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Solution by Dan Jurca

Let $\mathbf{n} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$; for each vector $\mathbf{v}$ in $\mathbf{R}^3$ we have the projection of $\mathbf{v}$ onto $\Pi$, the plane with equation $ax+by+cz=0$ normal to $\mathbf{n}$ and passing through the origin, by

$$\text{proj}_\Pi \mathbf{v} = \mathbf{v} - \frac{\mathbf{n}\cdot\mathbf{v}}{\|\mathbf{n}\|^2}\mathbf{n}.$$

With $r^2 = \|\mathbf{n}\|^2 = a^2+b^2+c^2$ and

$$\begin{aligned} \mathbf{x} &= \text{proj}_\Pi \mathbf{i} = \mathbf{i} - \frac{a}{r^2}\mathbf{n} \\ \mathbf{y} &= \text{proj}_\Pi \mathbf{j} = \mathbf{j} - \frac{b}{r^2}\mathbf{n} \\ \mathbf{z} &= \text{proj}_\Pi \mathbf{k} = \mathbf{k} - \frac{c}{r^2}\mathbf{n}; \end{aligned}$$

we interpret the problem to mean finding the angles $\theta_{\mathbf{xy}}$, $\theta_{\mathbf{yz}}$, and $\theta_{\mathbf{zx}}$, between the vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{y}$ and $\mathbf{z}$, and $\mathbf{z}$ and $\mathbf{x}$, respectively. To do this we compute

$$\begin{aligned}
\mathbf{x}\cdot\mathbf{y} &= \mathbf{i}\cdot\mathbf{j} - \frac{b}{r^2}\mathbf{i}\cdot\mathbf{n} - \frac{a}{r^2}\mathbf{n}\cdot\mathbf{j} + \frac{ab}{(r^2)^2}\mathbf{n}\cdot\mathbf{n} = \frac{-ba-ab+ab}{r^2} = -\frac{ab}{r^2} \\
\mathbf{y}\cdot\mathbf{z} &= \mathbf{j}\cdot\mathbf{k} - \frac{c}{r^2}\mathbf{j}\cdot\mathbf{n} - \frac{b}{r^2}\mathbf{n}\cdot\mathbf{k} + \frac{bc}{(r^2)^2}\mathbf{n}\cdot\mathbf{n} = \frac{-cb-bc+bc}{r^2} = -\frac{bc}{r^2} \\
\mathbf{z}\cdot\mathbf{x} &= \mathbf{k}\cdot\mathbf{i} - \frac{a}{r^2}\mathbf{k}\cdot\mathbf{n} - \frac{c}{r^2}\mathbf{n}\cdot\mathbf{i} + \frac{ca}{(r^2)^2}\mathbf{n}\cdot\mathbf{n} = \frac{-ac-ca+ca}{r^2} = -\frac{ca}{r^2} \\
\|\mathbf{x}\|^2 = \mathbf{x}\cdot\mathbf{x} &= \mathbf{i}\cdot\mathbf{i} - \frac{a}{r^2}\mathbf{i}\cdot\mathbf{n} - \frac{a}{r^2}\mathbf{n}\cdot\mathbf{i} + \frac{a^2}{(r^2)^2}\mathbf{n}\cdot\mathbf{n} = \frac{r^2}{r^2} - \frac{2a^2}{r^2} + \frac{a^2}{r^2} = \frac{r^2-a^2}{r^2} = \frac{b^2+c^2}{r^2} \\
\|\mathbf{y}\|^2 = \mathbf{y}\cdot\mathbf{y} &= \mathbf{j}\cdot\mathbf{j} - \frac{b}{r^2}\mathbf{j}\cdot\mathbf{n} - \frac{b}{r^2}\mathbf{n}\cdot\mathbf{j} + \frac{b^2}{(r^2)^2}\mathbf{n}\cdot\mathbf{n} = \frac{r^2}{r^2} - \frac{2b^2}{r^2} + \frac{b^2}{r^2} = \frac{r^2-b^2}{r^2} = \frac{c^2+a^2}{r^2} \\
\|\mathbf{z}\|^2 = \mathbf{z}\cdot\mathbf{z} &= \mathbf{k}\cdot\mathbf{k} - \frac{c}{r^2}\mathbf{k}\cdot\mathbf{n} - \frac{c}{r^2}\mathbf{n}\cdot\mathbf{k} + \frac{c^2}{(r^2)^2}\mathbf{n}\cdot\mathbf{n} = \frac{r^2}{r^2} - \frac{2c^2}{r^2} + \frac{c^2}{r^2} = \frac{r^2-c^2}{r^2} = \frac{a^2+b^2}{r^2}.
\end{aligned}$$

We have then, from $\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta_{\mathbf{xy}}$ if $\mathbf{x}\neq\mathbf{0}$ and $\mathbf{y}\neq\mathbf{0}$, *etc.*,

$$\theta_{\mathbf{xy}} \begin{cases} = \cos^{-1}\left(-\dfrac{ab}{\sqrt{(b^2+c^2)(c^2+a^2)}}\right) & \text{if } c\neq 0 \text{ or } (a\neq 0 \text{ and } b\neq 0) \\ \text{not defined} & \text{otherwise,} \end{cases}$$

$$\theta_{\mathbf{yz}} \begin{cases} = \cos^{-1}\left(-\dfrac{bc}{\sqrt{(c^2+a^2)(a^2+b^2)}}\right) & \text{if } a\neq 0 \text{ or } (c\neq 0 \text{ and } b\neq 0) \\ \text{not defined} & \text{otherwise, and} \end{cases}$$

$$\theta_{\mathbf{zx}} \begin{cases} = \cos^{-1}\left(-\dfrac{ca}{\sqrt{(a^2+b^2)(b^2+c^2)}}\right) & \text{if } b\neq 0 \text{ or } (a\neq 0 \text{ and } c\neq 0) \\ \text{not defined} & \text{otherwise.} \end{cases}$$

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Also solved by Jan van Delden (the Netherlands) and Grant Morgan