**Problem for 2009 November**

Proposed by Dan Jurca

If the statement is true, prove it; if the statement is not true, give a counterexample.

a. Suppose $f : \mathbf{R} \to \mathbf{R}$ and there exists $a \in \mathbf{R}$ such that $a \le x \;\Rightarrow\; 0 < f(x)$ and $a \le x \;\Rightarrow\; 0 < f'(x)$; then $\lim_{\infty} f = \infty$.

b. Suppose $f : \mathbf{R} \to \mathbf{R}$ and there exists $a \in \mathbf{R}$ such that $a \le x \;\Rightarrow\; 0 < f(x)$ and $a \le x \;\Rightarrow\; 0 < f''(x)$; then $\lim_{\infty} f = \infty$.

c. Suppose $f : \mathbf{R} \to \mathbf{R}$ and there exists $a \in \mathbf{R}$ such that $a \le x \;\Rightarrow\; 0 < f'(x)$ and $a \le x \;\Rightarrow\; 0 < f''(x)$; then $\lim_{\infty} f = \infty$.

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Solution by the proposer

a. The statement is false. For example, let $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = \pi/2 + \arctan(x)$.
Then $x \in \mathbf{R} \;\Rightarrow\; 0 < f(x)$ and $0 < f'(x) = 1/(x^2+1)$; however, $\lim_{\infty} f = \pi$.

b. The statement is false. For example, let $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = e^{-x}$.
Then $x \in \mathbf{R} \;\Rightarrow\; 0 < f(x) = f''(x)$; however, $\lim_{\infty} f = 0$.

c. The statement is true.

Proof.

First assume there exists some number $M \in \mathbf{R}$ such that $a \le x \;\Rightarrow\; f(x) \le M$, and let $B = \operatorname{lub}\{f(x) \mid a \le x\}$. Then $a \le x \;\Rightarrow\; f(x) \le B$, and we obtain a contradiction as follows. By the mean value theorem there exists $\xi \in (a, a+1)$ such that $f(a+1) - f(a) = f'(\xi)$. Since $a < \xi$ it follows that $0 < f'(\xi)$, so that $f(a) < f(a+1)$; hence $f(a) < B$. Since there exist values of $x$ greater than $a$ such that $f(x)$ is arbitrarily close to $B$, but less than or equal to $B$, it follows from the intermediate value theorem that there exists $x_0$ such that $a < x_0$ and $f(x_0) = (f(a)+B)/2$. Now $0 < f'(x_0)$ and we let

$$h = \frac{B - f(a)}{2f'(x_0)}.$$

Then $0 < h$ and there exists $\eta \in (x_0, x_0+h)$ such that $f(x_0+h) - f(x_0) = f'(\eta)h$. But since $0 < f''$ in $(a, \infty)$, it follows that $f' \nearrow$ in $(a, \infty)$, so that $f'(x_0) < f'(\eta)$. Hence $(B - f(a))/2 = f'(x_0)h < f'(\eta)h$, from which

$$B = (f(a)+B)/2 + (B - f(a))/2 = f(x_0) + (B - f(a))/2 < f(x_0) + f'(\eta)h = f(x_0+h),$$

whence $B < f(x_0+h)$, a contradiction. Therefore $f|_{[a,\infty)}$ is unbounded above and strictly increasing, so $\lim_{\infty} f = \infty$; *i.e.*,

$$M \in \mathbf{R} \;\Rightarrow\; \exists x_0 \in \mathbf{R} \text{ such that } x_0 \le x \;\Rightarrow\; M \le f(x).$$

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Solution of c. by Bojan Bašić (Serbia)

The statement is true. Let $f'(a) = D > 0$. Since $f''(x) > 0$ whenever $x \ge a$, one has $f'(x) \ge D$ whenever $x \ge a$. Therefore, $f'(x) - D \ge 0$ for all $x \ge a$, and it follows that $0 \le \int_a^x (f'(t) - D)\,\mathrm{d}t = f(x) - f(a) - Dx + Da$. One now has $f(x) \ge Dx + f(a) - Da$ whenever $x \ge a$, and therefore $\lim_{x\to\infty} f(x) \ge \lim_{x\to\infty} (Dx + f(a) - Da) = \infty$.

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Also solved by Massoud Malek and John M. Sayer