We shall represent an ellipse analytically, in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, or

$$x^2b^2 + y^2a^2 = a^2b^2. \tag{1}$$

Let $\langle x_0, y_0 \rangle$ be a point outside the ellipse. All the lines through $\langle x_0, y_0 \rangle$ (actually, all of them but $x = x_0$, which shall be treated separately) are of the form

$$y = y_0 + k(x - x_0), \tag{2}$$

where k varies. Therefore, if we want to find a common point(s) of a line through $\langle x_0, y_0 \rangle$ and the ellipse, it is enough to plug (2) into (1) and check for which values of k the resulting equation has only one solution in x (and the line $x = x_0$ is tangent only for $x_0 = \pm a$; we leave this case for the end, and further assume, when needed, $x_0^2 - a^2 \neq 0$).

$$x^2b^2 + (y_0^2 + 2y_0k(x - x_0) + k^2(x - x_0)^2)a^2 = a^2b^2$$

$$\underline{\underline{x^2b^2}} + (y_0^2 + \underline{2y_0kx} - 2y_0kx_0 + \underline{\underline{k^2x^2}} - \underline{2k^2xx_0} + k^2x_0^2)a^2 = a^2b^2$$

This equation has only one solution in x iff the discriminant, D, is equal to 0.

$$D = (a^{\cancel{2}}(2y_0k - 2k^2x_0))^2 - 4(b^2 + a^2k^2)(y_0^2\cancel{a^2} - 2y_0kx_0\cancel{a^2} + k^2x_0^2\cancel{a^2} - \cancel{a^2}b^2) = 0$$

$$a^2(\cancel{4y_0^2k^2} - \cancel{8y_0k^3x_0} + \cancel{4k^4x_0^2}) - (4b^2y_0^2 - 8b^2y_0kx_0 + 4b^2k^2x_0^2 - 4b^4 + \cancel{4a^2k^2y_0^2} - \cancel{8a^2y_0k^3x_0} + \cancel{4a^2k^4x_0^2} - 4a^2b^2k^2) = 0$$

$$(\cancel{4b^2}x_0^2 - \cancel{4}a^2\cancel{b^2})k^2 - \cancel{8b^2}^{\,2}x_0y_0k + \cancel{4b^2}y_0^2 - \cancel{4b^4}^{\,2} = 0 \tag{3}$$

$$k_{1/2} = \frac{\cancel{2}x_0y_0 \pm \sqrt{\cancel{4}x_0^2y_0^2 - \cancel{4}(x_0^2 - a^2)(y_0^2 - b^2)}}{\cancel{2}(x_0^2 - a^2)} = \frac{x_0y_0 \pm \sqrt{x_0^2b^2 + y_0^2a^2 - a^2b^2}}{x_0^2 - a^2} \tag{4}$$

Let us now take a break and check what is done so far. In the first place, there are two values for k (and, therefore, two tangents) iff the expression under the $\sqrt{\ }$ is positive, what happens iff point $\langle x_0, y_0 \rangle$ is outside the ellipse. This proves part a. of the problem. For the part b., we recall that lines with coefficients k_1 and k_2 are orthogonal to each other iff $k_2 = -\dfrac{1}{k_1}$. Therefore, we impose this constraint on (4).

$$\frac{x_0y_0 - \sqrt{x_0^2b^2 + y_0^2a^2 - a^2b^2}}{x_0^2 - a^2} = -\frac{x_0^2 - a^2}{x_0y_0 + \sqrt{x_0^2b^2 + y_0^2a^2 - a^2b^2}}$$

$$\frac{x_0^2y_0^2 - (x_0^2b^2 + y_0^2a^2 - a^2b^2) + (x_0^2 - a^2)^2}{(x_0^2 - a^2)(x_0y_0 + \sqrt{x_0^2b^2 + y_0^2a^2 - a^2b^2})} = 0$$

$$(x_0^2y_0^2 - x_0^2b^2 - y_0^2a^2 + a^2b^2) + (x_0^2 - a^2)^2 = 0$$

$$(x_0^2 - a^2)(y_0^2 - b^2 + x_0^2 - a^2) = 0$$

$$x_0^2 + y_0^2 = a^2 + b^2 = \text{const}$$

(The possibility $x_0y_0 + \sqrt{x_0^2b^2 + y_0^2a^2 - a^2b^2} = 0$ has also been excluded and left for the end.)

Therefore, all the points in question, with the possible exception of the mentioned special cases, are on the same circle. Let us now deal with the remaining cases.

If $x_0 = \pm a$, then one tangent is the line $x = x_0$, while the other is the one with the corresponding $k = \dfrac{y_0^2 - b^2}{2x_0y_0}$ (this is gotten from (3); we are sure that $y_0 \neq 0$ since $\langle x_0, y_0 \rangle$ is outside the ellipse). Therefore, part a. is correct in this case, too. Furthermore, the second tangent is orthogonal to the first one iff $k = 0$, what is iff $y_0 = \pm b$. Therefore, $x_0^2 + y_0^2 = a^2 + b^2$ again.

If $x_0y_0 + \sqrt{x_0^2b^2 + y_0^2a^2 - a^2b^2} = 0$, then $0 = x_0^2y_0^2 - x_0^2b^2 - y_0^2a^2 + a^2b^2 = (x_0^2 - a^2)(y_0^2 - b^2)$, and therefore either $x_0 = \pm a$ (what is the previous case), or $y_0 = \pm b$ (what is the previous case backwards).

The circle is completed.

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