Problem for 2010 February

Proposed by Dan Jurca

Definition. Suppose $a,\ b \in \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ is continuous. We consider the sequence $(A_n)_{n=0}^{\infty}$ as follows.

$$A_0 = b \qquad \text{and} \qquad 1 \le n \ \Rightarrow\ A_n = \int_a^{A_{n-1}} f$$

Then if $(A_n)_{n=0}^{\infty}$ converges to $A \in \{-\infty\} \cup \mathbf{R} \cup \{\infty\}$, we write $\displaystyle\int_a^{b\nearrow} f = A$.

a. Show that if $A \in \{-\infty\} \cup \mathbf{R} \cup \{\infty\}$, then there exists continuous $f : \mathbf{R} \to \mathbf{R}$ such that $\displaystyle\int_0^{1\nearrow} f = A$.

b. Show that if $a,\ b \in \mathbf{R}$ and $a \ne 0$, then there does not exist continuous $f : \mathbf{R} \to \mathbf{R}$ such that $\displaystyle\int_a^{b\nearrow} f = a$.

Solution of a. by Bojan Bašić (Serbia)

a. Let $A \in \mathbf{R}$. Choose $f(x) = -2x + A + 1$. Then

$$A_1 = \int_0^1 (-2x + A + 1)\,\mathrm{d}x = (-x^2 + Ax + x)\Big|_0^1 = (-1 + A + 1) - 0 = A;$$
$$A_2 = \int_0^A (-2x + A + 1)\,\mathrm{d}x = (-x^2 + Ax + x)\Big|_0^A = (-A^2 + A^2 + A) - 0 = A = A_1;$$

and it is now obvious that $\displaystyle\int_0^{1\nearrow} f = A$.

If $A = \infty$, it is enough to take $f(x) \equiv 2$. If $A = -\infty$, $f(x) = -4x$ does the job.

Solution of b. by the proposer

b. Assuming otherwise, we let $F(x) = \displaystyle\int_a^x f$. Then $F(a) = 0$, $F' = f$, and we have the sequence $(A_n)_{n=0}^{\infty}$ where

$$A_0 = b \qquad \text{and} \qquad 1 \le n \ \Rightarrow\ A_n = \int_a^{A_{n-1}} f = F(A_{n-1}) - F(a) = F(A_{n-1}) - 0 = F(A_{n-1}).$$

Then $a = \lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} F(A_{n-1}) = F\left(\lim\limits_{n\to\infty} A_{n-1}\right) = F(a) = 0$, an obvious contradiction. Therefore no such f exists.

Part b. also solved by Bojan Bašić (Serbia).