Problem for 2010 March

Communicated by Dan Jurca

Determine all integers n such that n equals the sum of two or more consecutive positive integers.

Solution by Dan Jurca

Proposition. The positive integer n equals the sum of two or more consecutive positive integers if and only if n does not equal a power of 2.

Proof.

First, if $0 \leq k,\ 2 \leq m$, and $n = (k+1) + (k+2) + \cdots + (k+m)$, then $n = mk + m(m+1)/2$, so that $2n = 2mk + m(m+1) = m(2k+m+1)$. If m equals an even integer, then $2k+m+1$ equals an odd integer; in any case there exists an odd integer greater than 1 which divides $2n$, hence which divides n; it follows that n does not equal a power of 2.

Next, suppose n does not equal a power of 2; then there exist a unique nonnegative integer a and a unique positive integer b such that $n = 2^a(2b+1)$. If $2^a \leq b$, let $k = b - 2^a$; then $0 \leq k$ and

$$\begin{aligned}
(k+1) + (k+2) + \cdots + (k+2^{a+1}) &= 2^{a+1}k + (1 + 2 + \cdots + 2^{a+1}) \\
&= 2^{a+1}k + \frac{2^{a+1}(2^{a+1}+1)}{2} \\
&= 2^{a+1}k + 2^a(2^{a+1}+1) \\
&= 2^a(2k + 2^{a+1} + 1) \\
&= 2^a[2(b - 2^a) + 2^{a+1} + 1] \\
&= 2^a(2b - 2^{a+1} + 2^{a+1} + 1) \\
&= 2^a(2b+1) \\
&= n,
\end{aligned}$$

so that n equals the sum of 2^{a+1} consecutive positive integers. If $b < 2^a$, let $k = 2^a - b - 1$; then $0 \leq k$ and

$$\begin{aligned}
(k+1) + (k+2) + \cdots + [k + (2b+1)] &= (2b+1)k + [1 + 2 + \cdots + (2b+1)] \\
&= (2b+1)k + \frac{(2b+1)(2b+2)}{2} \\
&= (2b+1)k + (2b+1)(b+1) \\
&= (2b+1)[k + (b+1)] \\
&= (2b+1)[(2^a - b - 1) + b + 1] \\
&= (2b+1) \cdot 2^a \\
&= 2^a(2b+1) \\
&= n,
\end{aligned}$$

so that n equals the sum of $2b+1$ consecutive positive integers.

Also solved by Bojan Bašić (Serbia), Massoud Malek, Bill Nico, and John M. Sayer