Problem for 2010 April

Proposed by Dan Jurca

Let q_n equal the product of the first n prime numbers. ($q_1 = 2,\ q_2 = 6,\ q_3 = 30,\ etc.$) Prove $\displaystyle\sum_{n=1}^{\infty} \frac{1}{q_n} < \frac{1}{\sqrt{2}}$.

Solution by the proposer

For $n = 1,\ 2,\ \ldots$ let p_n equal the n-th prime. ($p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ etc.$) Then for each $N,\ 1 \le N$,

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{q_n} < \sum_{n=1}^{\infty} \frac{1}{q_n} &= \sum_{n=1}^{N} \frac{1}{q_n} + \sum_{n=N+1}^{\infty} \frac{1}{q_n} \\
&= \sum_{n=1}^{N} \frac{1}{q_n} + \left[\frac{1}{q_{N+1}} + \frac{1}{q_{N+2}} + \frac{1}{q_{N+3}} + \frac{1}{q_{N+4}} + \cdots\right] \\
&= \sum_{n=1}^{N} \frac{1}{q_n} + \frac{1}{q_{N+1}}\left[1 + \frac{1}{p_{N+2}} + \frac{1}{p_{N+2}p_{N+3}} + \frac{1}{p_{N+2}p_{N+3}p_{N+4}} + \cdots\right] \\
&< \sum_{n=1}^{N} \frac{1}{q_n} + \frac{1}{q_{N+1}}\left[1 + \frac{1}{p_{N+2}} + \frac{1}{(p_{N+2})^2} + \frac{1}{(p_{N+2})^3} + \cdots\right] \\
&= \sum_{n=1}^{N} \frac{1}{q_n} + \frac{1}{q_{N+1}} \cdot \frac{1}{1 - 1/p_{N+2}} \\
&= \sum_{n=1}^{N} \frac{1}{q_n} + \frac{1}{q_{N+1}} \cdot \frac{p_{N+2}}{p_{N+2} - 1}; \quad \text{hence}
\end{aligned}$$

$$\sum_{n=1}^{N} \frac{1}{q_n} < \sum_{n=1}^{\infty} \frac{1}{q_n} < \sum_{n=1}^{N} \frac{1}{q_n} + \frac{p_{N+2}}{p_{N+2} - 1} \cdot \frac{1}{q_{N+1}}.$$

With $N = 3$ we have

$$\sum_{n=1}^{3} \frac{1}{q_n} < \sum_{n=1}^{\infty} \frac{1}{q_n} < \sum_{n=1}^{3} \frac{1}{q_n} + \frac{p_5}{p_5 - 1} \cdot \frac{1}{q_4}, \quad \text{so}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{30} < \sum_{n=1}^{\infty} \frac{1}{q_n} < \frac{1}{2} + \frac{1}{6} + \frac{1}{30} + \frac{11}{10} \cdot \frac{1}{210}, \quad \text{whence}$$

$$\frac{7}{10} < \sum_{n=1}^{\infty} \frac{1}{q_n} < \frac{7}{10} + \frac{11}{2{,}100}, \quad \text{or}$$

$$\frac{7}{10} < \sum_{n=1}^{\infty} \frac{1}{q_n} < \frac{1{,}481}{2{,}100}; \quad \text{and since}$$

$$\begin{aligned}
\left(\frac{1{,}481}{2{,}100}\right)^2 &= \frac{2{,}193{,}361}{4{,}410{,}000} \\
&< \frac{1}{2}
\end{aligned}$$

the asserted inequality follows.

In fact $\displaystyle\sum_{n=1}^{\infty} \frac{1}{q_n} = 0.7052301717918\ldots < 0.705230171792 < 0.7071067811865 < 0.70710678118654\ldots = \frac{1}{\sqrt{2}}$.

Also solved by Bojan Bašić (Serbia) and Massoud Malek