Problem for 2010 May

Communicated by Dan Jurca

a. Find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that
   $x \in \mathbf{R} \;\Rightarrow\; f(f(f(x))) = x$.

b. Find a function $f : \mathbf{R} \to \mathbf{R}$ such that
   i. $x \in \mathbf{R} \;\Rightarrow\; f(f(f(x))) = x$; and
   ii. $x \in \mathbf{R} \;\Rightarrow\; f(x) \neq x$.

Solution by Dan Jurca

a. Proposition. If $f : \mathbf{R} \to \mathbf{R}$ is continuous and $x \in \mathbf{R} \;\Rightarrow\; f(f(f(x))) = x$, then $f = \mathrm{id}_{\mathbf{R}}$.

Proof.

Since $f \circ (f \circ f) = \mathrm{id}_{\mathbf{R}} = (f \circ f) \circ f$, it follows that f is a homeomorphism (with inverse $f \circ f$); in particular, f is injective. Now suppose $a \in \mathbf{R}$ and $f(a) \neq a$; we derive a contradiction as follows.

Let $b = f(a)$ and $c = f(b)$. Since $b = f(a) \neq a$, it follows that $b \neq a$, and hence $c \neq b$. Also $c \neq a$, since otherwise $f(c) = f(f(b)) = f(f(f(a))) = a$, so $f(a) = a$, a contradiction. Thus $a \neq b$, $b \neq c$, and $c \neq a$. We next consider the following two possibilities: $f(a) < a$ or $a < f(a)$. Recall that $f(c) = f(f(b)) = f(f(f(a))) = a$.

If $f(a) < a$, then $b < a$. Then $c < b < a$ or $b < c < a$ or $b < a < c$. In the first case we have $f(b) < f(a) < f(c)$, so by the intermediate value theorem (and continuity of f) there exists $\xi \in (c, b)$ such that $f(\xi) = f(a)$. But then $\xi \neq c$, contradicting injectivity of f. In the second case we have $f(a) < f(b) < f(c)$, so again there exists $\xi \in (c, a)$ such that $f(\xi) = f(b)$, another contradiction. In the third case we have $f(a) < f(c) < f(b)$, so once again there exists $\xi \in (b, a)$ such that $f(\xi) = f(c)$, yet another contradiction. Therefore $a \leq f(a)$.

The argument in case $a < f(a)$ is similar. Since there does not exist $a \in \mathbf{R}$ such that $f(a) \neq a$, it follows that $x \in \mathbf{R} \;\Rightarrow\; f(x) = x$, so $f = \mathrm{id}_{\mathbf{R}}$.

b. Recall the notation: $\{\ \} : \mathbf{R} \to \mathbf{R}$ by $x \mapsto \{x\} = x - \lfloor x \rfloor$; *i.e.*, $\{x\}$ is the *fractional part* of x. Then consider $f : \mathbf{R} \to \mathbf{R}$ by $x \in \mathbf{R} \;\Rightarrow\; f(x) = \lfloor x \rfloor + \{x + 1/3\}$. Suppose $x \in \mathbf{R}$, and say $x = n + \theta$, where n is an integer and $0 \leq \theta < 1$, so $\lfloor x \rfloor = n$ and $\{x\} = \theta$. (n and θ are clearly uniquely determined.) Then $f(x) = n + \{\theta + 1/3\}$, $f(f(x)) = n + \{\theta + 2/3\}$, and $f(f(f(x))) = n + \{\theta + 1\} = n + \theta = x$, so $f(f(f(x))) = x$. Since $\forall \theta :\ \{\theta + 1/3\} \neq \{\theta\}$, there exists no fixed point of f.