Problem for 2010 June

Proposed by Dan Jurca

Suppose $r \in \mathbf{R}$, $0 < r$, and for $n = 1, 2, 3, \ldots$

$$x_n = \sqrt[r]{1} + \sqrt[r]{2} + \sqrt[r]{3} + \cdots + \sqrt[r]{n}.$$

Show that

$$\sum_{n=1}^{\infty} \frac{1}{x_n} < \infty.$$

Solution by the proposer

The function $f : [0, \infty) \to \mathbf{R}$ by $f(t) = \sqrt[r]{t} = t^{1/r}$ strictly increases, so that

$$1 \leq n \text{ and } t \in [n-1, n] \;\Rightarrow\; t^{1/r} \leq n^{1/r}; \text{ hence}$$

$$\int_{n-1}^{n} t^{1/r}\,dt < n^{1/r}, \text{ so}$$

$$\int_{0}^{n} t^{1/r}\,dt < \sqrt[r]{1} + \sqrt[r]{2} + \cdots + \sqrt[r]{n}; \text{ thus}$$

$$\frac{r}{r+1} n^{1+1/r} < x_n, \quad \text{so}$$

$$\frac{1}{x_n} < \frac{r+1}{r} \frac{1}{n^{1+1/r}},$$

whence by comparison with the $(1 + 1/r)$-series, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x_n} < \infty$.

Remark. The argument shows that

$$\sum_{n=1}^{\infty} \frac{1}{x_n} < \frac{r+1}{r} \zeta(1 + 1/r)$$

where ζ equals the Riemann zeta function.