Problem for 2010 September and October

Proposed by Dan Jurca

Consider the equation $x^3 + 2y^3 + 4z^3 = 6xyz$.

a. Prove that there exists no non-trivial integer solution of the equation; *i.e.*, if $x \in \mathbf{Z},\ y \in \mathbf{Z},\ z \in \mathbf{Z}$, and $x^3 + 2y^3 + 4z^3 = 6xyz$, then $x = y = z = 0$.

b. Prove that there exists no non-trivial rational solution of the equation; *i.e.*, if $x \in \mathbf{Q},\ y \in \mathbf{Q},\ z \in \mathbf{Q}$, and $x^3 + 2y^3 + 4z^3 = 6xyz$, then $x = y = z = 0$.

Solution by the proposer

a. If each of x, y, and z is an integer, $x^3 + 2y^3 + 4z^3 = 6xyz$, and some integer t divides each of x, y, and z, then say $x = tu$, $y = tv$, and $z = tw$. Then $(tu)^3 + 2(tv)^3 + 4(tw)^3 = 6(tu)(tv)(tw)$, so that $u^3 + 2v^3 + 4w^3 = 6uvw$. Therefore if a non-trivial solution in integers exists, then there exists a solution (x, y, z) where $\gcd(x, y, z) = 1$, in particular at least one of x, y, and z is odd. Obviously x must be even, say $x = 2u$. Then $8u^3 + 2y^3 + 4z^3 = 12uyz$, so that $4u^3 + y^3 + 2z^3 = 6uyz$. Therefore y is even, say $y = 2v$. But then $4u^3 + 8v^3 + 2z^3 = 12uvz$, so that $2u^3 + 4v^3 + z^3 = 6uvz$. Hence z is even. This contradiction shows that the equation has no nontrivial integer solution.

b. Assume $a \in \mathbf{Z},\ b \in \mathbf{Z},\ c \in \mathbf{Z},\ d \in \mathbf{Z},\ e \in \mathbf{Z},\ f \in \mathbf{Z}$, and $b \neq 0,\ d \neq 0,\ f \neq 0$, and $x = a/b,\ y = c/d,\ z = e/f$, and $x^3 + 2y^3 + 4z^3 = 6xyz$. Then follows

$$\frac{a^3}{b^3} + 2\frac{c^3}{d^3} + 4\frac{e^3}{f^3} = 6\frac{a}{b}\frac{c}{d}\frac{e}{f} \quad \text{so}$$
$$a^3d^3f^3 + 2b^3c^3f^3 + 4b^3d^3e^3 = 6ab^2cd^2ef^2 \quad \text{or}$$
$$(adf)^3 + 2(bcf)^3 + 4(bde)^3 = 6(adf)(bcf)(bde).$$

But then if $u = adf,\ v = bcf$, and $w = bde$, we have $u \in \mathbf{Z},\ v \in \mathbf{Z},\ w \in \mathbf{Z}$, and $u^3+2v^3+4w^3 = 6uvw$. Therefore by the result in part a. we have $u = 0,\ v = 0$, and $w = 0$, so that $a = 0,\ c = 0$, and $e = 0$, whence $x = y = z = 0$.

Also solved by Bojan Bašić (Serbia), Matthew Felix, Nick Grener, Massoud Malek, John Sayer, and Winston Teitler