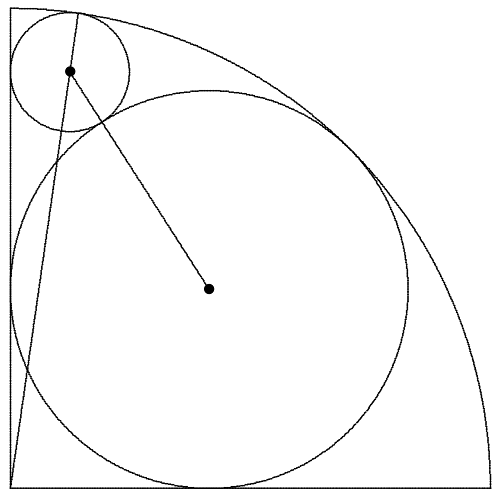
Problem for 2010 December

Communicated by Dan Jurca



The figure above shows one fourth of a circle of radius 1 centered at the origin of coordinates, a second circle tangent to this first circle and the positive coordinate axes, and finally a third circle tangent to the other two circles and the positive y-axis. What is the radius of this third circle?

Solution by Dan Jurca

Suppose the radius of the third circle is r, the y-coordinate of the center is s, and the radius of the second circle is t. Then the center of the third circle is at (r, s), the center of the second circle is at (t, t), and we have the following equations.

$$\sqrt{2}\,t + t = 1$$
$$\sqrt{r^2 + s^2} + r = 1$$
$$(t - r)^2 + (t - s)^2 = (r + t)^2$$

Hence $t = 1/(\sqrt{2}+1) = \sqrt{2}-1$, $s = \sqrt{1 - 2r}$, and we have from the third equation $-2tr+t^2-2ts+s^2 = 2tr$. Eliminating s, we find the following.

$$(9 - 4\sqrt{2})r^2 + (18 - 14\sqrt{2})r + (3 - 2\sqrt{2}) = 0$$

Since one of the roots of this equation equals $t = \sqrt{2} - 1$ and the product of the two roots equals

$$\frac{3 - 2\sqrt{2}}{9 - 4\sqrt{2}}$$

we find that

$$r = \frac{3 - 2\sqrt{2}}{(9 - 4\sqrt{2})(\sqrt{2} - 1)} = -\frac{3 - 2\sqrt{2}}{17 - 13\sqrt{2}} = -\frac{3 - 2\sqrt{2}}{17 - 13\sqrt{2}} \cdot \frac{17 + 13\sqrt{2}}{17 + 13\sqrt{2}} = \frac{1 - 5\sqrt{2}}{289 - 338} = \frac{1 - 5\sqrt{2}}{-49} = \frac{\sqrt{50} - 1}{50 - 1}.$$

Also solved by Matthew Felix, Massoud Malek, Winston Teitler, and two others whose names and solution I have lost (sorry — please resubmit)