Problem for 2011 January

Communicated by Dan Jurca

Prove that if $p(x)$ is a real polynomial, not identically zero, and $x \in \mathbf{R} \;\Rightarrow\; 0 \le p(x)$, then the function $f : \mathbf{R} \to \mathbf{R}$ by
$$f(x) = p(x) + p'(x) + p''(x) + p'''(x) + \cdots$$
is strictly positive.

Solution by Dan Jurca

We show that
$$x \in \mathbf{R} \;\Rightarrow\; f(x) = e^x \int_x^\infty e^{-t} p(t)\, dt,$$
from which the assertion follows.

First we observe that if $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$, then
$$f(0) = \sum_{i=0}^n c_i i!$$
and next that $f' = p' + p'' + p''' + \cdots = f - p$; hence $f' - f = -p$. Multiplying the differential equation $y' - y = -p(x)$ by the integrating factor e^{-x} we have $\left(e^{-x}y\right)' = -e^{-x}p(x)$, from which we find that a solution of the equation is
$$y = y(x) = e^x \int_x^\infty e^{-t} p(t)\, dt$$
and since
$$e^0 \int_0^\infty e^{-t} p(t)\, dt = \int_0^\infty e^{-t} \sum_{i=0}^n c_i t^i\, dt = \sum_{i=0}^n c_i \int_0^\infty e^{-t} t^i\, dt = \sum_{i=0}^n c_i i! = f(0),$$
it follows that
$$f(x) = e^x \int_x^\infty e^{-t} p(t)\, dt,$$
as asserted.

Finally since e^x is positive and since p vanishes at only finitely many points (if any), the integral of $e^{-t}p(t)$ over each nonempty finite interval is positive; therefore f is a strictly positive function.

Solution by Massoud Malek

Let $g(x) = f(x)e^{-x}$; then
$$g'(x) = f'(x)e^{-x} - f(x)e^{-x} = [f'(x) - f(x)]e^{-x} = -p(x)e^{-x}.$$
Since $0 \le p(x)$ and e^{-x} is always positive, we conclude that g is a decreasing function. Since also
$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{f(x)}{e^x} = 0$$
and e^{-x} is positive for each x, it follows that $0 < f(x)$ for each $x \in \mathbf{R}$.