Problem for 2011 March

Communicated by Dan Jurca

Suppose $0 < x < y$; determine the greater of the following.

$$\frac{1 + x + x^2 + \cdots + x^{100}}{1 + x + x^2 + \cdots + x^{100} + x^{101}}$$

$$\frac{1 + y + y^2 + \cdots + y^{100}}{1 + y + y^2 + \cdots + y^{100} + y^{101}}$$

Solution by Bill Nico

For each nonnegative integer n let $s_n : \mathbf{R} \to \mathbf{R}$ by $s_n(t) = 1 + t + t^2 + \cdots + t^n$ and let $f_n : [0, \infty) \to \mathbf{R}$ by

$$\begin{aligned}
f_n(t) &= \frac{s_n(t)}{s_n(t) + t^{n+1}}. \quad \text{Then} \\
0 < t \;\Rightarrow\; f_n'(t) &= \frac{s_n'(t)\left(s_n(t) + t^{n+1}\right) - s_n(t)\left(s_n'(t) + (n+1)t^n\right)}{\left(s_n(t) + t^{n+1}\right)^2}, \quad \text{so} \\
0 < t \;\Rightarrow\; \left(s_n(t) + t^{n+1}\right)^2 f_n'(t) &= s_n'(t)t^{n+1} - (n+1)s_n(t)t^n \\
&= t^n\left(ts_n'(t) - (n+1)s_n(t)\right) \\
&= t^n\left[\sum_{i=1}^{n} it^i - (n+1)\left(1 + \sum_{i=1}^{n} t^i\right)\right] \\
&= t^n\left[\left(\sum_{i=1}^{n}(i - (n+1))t^i\right) - (n+1)\right] \\
&< 0.
\end{aligned}$$

Thus f_n decreases in $[0, \infty)$, so that $0 < x < y \;\Rightarrow\; f_n(y) < f_n(x)$; therefore, and in particular, we have $0 < x < y \;\Rightarrow\; f_{100}(y) < f_{100}(x)$.

Also solved by Matthew Felix, Massoud Malek, John M. Sayer, and Winston Teitler