Problem for 2011 April

Communicated by Dan Jurca

a. Show that the circle centered at the origin (0,0) with radius $\sqrt{3}$ contains no point (x,y) such that each of x and y is rational; *i.e.*, there do not exist rational numbers x and y such that $x^2 + y^2 = 3$.

b. Show that there exist infinitely many pairs of integers (x,y) such that $x^4 + x^2y - y^3 = 0$.

Solution of part a. by Massoud Malek

Suppose x and y are rational numbers such that $x^2+y^2 = 3$; then there are integers $p,\ q,\ r$, and s such that $\gcd(p,q) = \gcd(r,s) = 1,\ x = p/q,\ y = r/s$, and $p^2/q^2 + r^2/s^2 = 3$. Thus

$$\frac{r^2}{s^2} = 3 - \frac{p^2}{q^2} = \frac{3q^2 - p^2}{q^2},$$

so $3q^2 - p^2 = q^2r^2/s^2$, an integer, say v^2; thus $p^2 + v^2 = 3q^2$. It follows that $p \equiv 0 \pmod 3$ and $v \equiv 0 \pmod 3$, (since if n is an integer and $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$, then $n^2 \equiv 1 \pmod 3$) say $p = 3p_1$ for some integer p_1, and $v = 3v_1$ for some integer v_1. Hence $9p_1^2 + 9v_1^2 = 3q^2$, so that $q^2 = 3(p_1^2 + v_1^2)$ and thus $3|p$ and $3|q$, contradicting $\gcd(p,q) = 1$. Therefore no such x and y exist.

Solution of part b. by Dan Jurca

For each nonnegative integer n let

$$\begin{aligned} x_n &= n^3 + 3n^2 + 2n \\ &= n(n+1)(n+2) \qquad \text{and} \\ y_n &= n^4 + 4n^3 + 5n^2 + 2n \\ &= n(n+1)^2(n+2). \end{aligned}$$

Then the pairs of integers (x_n,y_n) are distinct and $0 \le n \ \Rightarrow$

$$\begin{aligned} x_n^4 + x_n^2 y_n - y_n^3 &= n^4(n+1)^4(n+2)^4 + n^2(n+1)^2(n+2)^2 \cdot n(n+1)^2(n+2) - n^3(n+1)^6(n+2)^3 \\ &= n^4(n+1)^4(n+2)^4 + n^3(n+1)^4(n+2)^3 - n^3(n+1)^6(n+2)^3 \\ &= n^3(n+1)^4(n+2)^3\left[n(n+2) + 1 - (n+1)^2\right] \\ &= n^3(n+1)^4(n+2)^3\left[n^2 + 2n + 1 - (n+1)^2\right] \\ &= 0. \end{aligned}$$

Also solved by Matthew Felix, Kourosh Ghaderi, Massoud Malek, Bill Nico, John M. Sayer, Winston Teitler, and Jan van Delden (the Netherlands)