Problem for 2011 July

Proposed by Dan Jurca

Determine the following limit.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$$

Solution by the proposer

We find $1 \le x \;\Rightarrow$

$$\begin{aligned}
\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right) &= \left(\sqrt{x^2+x}\right)^2-\left(\sqrt{x^2-x}\right)^2\\
&= \left(x^2+x\right)-\left(x^2-x\right)\\
&= 2x, \quad \text{so that}\\
\sqrt{x^2+x}-\sqrt{x^2-x} &= \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}\\
&= \frac{2}{\sqrt{1+1/x}+\sqrt{1-1/x}};\\
\text{whence}\quad \lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right) &= \lim_{x\to\infty}\frac{2}{\sqrt{1+1/x}+\sqrt{1-1/x}}\\
&= \frac{2}{1+1}\\
&= 1.
\end{aligned}$$

Also solved by Kouroush Ghaderi, Massoud Malek, John M. Sayer, Winston Teitler, and Jan van Delden (the Netherlands)