Problem for 2011 August, September, and October

Proposed by Dan Jurca

For each integer $n,\ 3 \le n$, let

$p_n =$ the perimeter of the regular n-gon inscribed in the unit circle,
$q_n =$ the perimeter of the regular n-gon circumscribed about the unit circle,
$A_n =$ the area of the regular n-gon inscribed in the unit circle, and
$B_n =$ the area of the regular n-gon circumscribed about the unit circle.

Show: if $3 \le n$, then p_n more nearly approximates the perimeter of the circle than does q_n; but if $4 \le n$, then B_n more nearly approximates the area of the circle than does A_n. That is, prove

$$3 \le n \ \Rightarrow\ |2\pi - p_n| < |2\pi - q_n| \qquad \text{and} \qquad 4 \le n \ \Rightarrow\ |\pi - B_n| < |\pi - A_n|.$$

Solution by the proposer

We find easily that

$$3 \le n \ \Rightarrow\ p_n = 2n\sin\frac{\pi}{n},\ q_n = 2n\tan\frac{\pi}{n},\ A_n = n\sin\frac{\pi}{n}\cos\frac{\pi}{n},\ B_n = n\tan\frac{\pi}{n},$$

and that $3 \le n \ \Rightarrow\ p_n < 2\pi < q_n$ and $A_n < \pi < B_n$.

If $f(\theta) = \sin\theta + \tan\theta - 2\theta$, then $f(0) = 0$, $f'(\theta) = \cos\theta + \sec^2\theta - 2$, and $f''(\theta) = -\sin\theta + 2\sec^2\theta\tan\theta = \sin\theta(2\sec^3\theta - 1)$. With $\varphi(t) = 2\sec^3 t - 1$ we have $\varphi'(t) = 6\sec^2 t\tan t$, so $0 < t < \pi/2 \ \Rightarrow\ 0 < \varphi'(t)$. Hence $0 < \theta < \pi/2 \ \Rightarrow\ 0 < f''(\theta)$, so $f' \nearrow$ in $[0, \pi/2]$. Since $f'(0) = 0$, it follows that f increases in $[0, \pi/2]$. Therefore $0 < \theta \le \pi/3 \ \Rightarrow\ 0 < f(\theta)$, so that $3 \le n \ \Rightarrow\ 0 < \pi/n \le \pi/3$ and therefore $0 < \sin(\pi/n) + \tan(\pi/n) - 2(\pi/n)$ so that $0 < n\sin(\pi/n) + n\tan(\pi/n) - 2\pi$. Thus $3 \le n \ \Rightarrow\ 2\pi - 2n\sin(\pi/n) < 2n\tan(\pi/n) - 2\pi$, so $2\pi - p_n < q_n - 2\pi$.

If $g : (-\pi/2, \pi/2) \to \mathbf{R}$ by $g(\theta) = 2\theta - \sin\theta\cos\theta - \tan\theta$, then $g'(\theta) = 2 - \cos^2\theta + \sin^2\theta - \sec^2\theta = 3 - 2\cos^2\theta - \sec^2\theta$. We observe that $g'(0) = 0$. Next, $g''(\theta) = 4\cos\theta\sin\theta - 2\sec^2\theta\tan\theta = 4\sin\theta\cos\theta - 2\sin\theta/\cos^3\theta = 2\sin\theta(2\cos\theta - \cos^{-3}\theta)$. If $\psi : (-\pi/2, \pi/2) \to \mathbf{R}$ by $\psi(\theta) = 2\cos\theta - \cos^{-3}\theta$, then $\psi'(\theta) = -2\sin\theta + 3\cos^{-4}\theta \cdot -\sin\theta = -(2\sin\theta + 3\sin\theta\cos^{-4}\theta)$. It follows that $0 < \theta < \pi/2 \ \Rightarrow\ \psi'(\theta) < 0$, so that ψ decreases in $[0, \pi/2)$. Now $\psi(0) = 1$, and $\psi(\pi/6) = 2 \cdot \sqrt{3}/2 - 1/(\sqrt{3}/2)^3 = \sqrt{3} - 8/(3\sqrt{3}) = 9\sqrt{3}/9 - 8\sqrt{3}/9 = \sqrt{3}/9 > 0$. Therefore $0 \le \theta \le \pi/6 \ \Rightarrow\ 0 < \psi(\theta)$. Since $g''(0) = 0$ it follows that $0 < \theta \le \pi/6 \ \Rightarrow\ 0 < g''(\theta)$, so g' increases in $[0, \pi/6)$. Since $g'(0) = 0$, it follows that g increases in $[0, \pi/6)$. Hence $0 \le \theta \le \pi/6 \ \Rightarrow\ 0 < g(\theta)$. Thus $6 \le n \ \Rightarrow\ 0 < 2 \cdot \pi/n - \sin\pi/n \cdot \cos\pi/n - \tan\pi/n$, so that $\tan\pi/n - \pi/n < \pi/n - \sin\pi/n \cdot \cos\pi/n$, whence $n\tan\pi/n - \pi < \pi - n\sin\pi/n \cdot \cos\pi/n$ from which $6 \le n \ \Rightarrow\ B_n - \pi < \pi - A_n$. By straightforward computation $B_4 - \pi < \pi - A_4$ and $B_5 - \pi < \pi - A_5$. Therefore, and finally, $4 \le n \ \Rightarrow\ B_n - \pi < \pi - A_n$.

Also solved by Kourosh Ghaderi and Winston Teitler