Problem for 2011 November

Communicated by Dan Jurca

For $2 \le n$ let $A_n = \prod_{k=2}^{n} \frac{k^3+1}{k^3-1}$; find $\lim_{n\to\infty} A_n$.

Solution by Dan Jurca

First we show $2 \le n \;\Rightarrow\; A_n = \frac{3}{2} \cdot \frac{n^3-n}{n^3-1}$.

By definition $A_2 = \frac{2^3+1}{2^3-1} = \frac{9}{7} = \frac{3}{2} \cdot \frac{6}{7} = \frac{3}{2} \cdot \frac{2^3-2}{2^3-1}$.

Next, if $3 \le n$ and

$$\begin{aligned} A_{n-1} &= \frac{3}{2} \cdot \frac{(n-1)^3-(n-1)}{(n-1)^3-1} \\ &= \frac{3}{2} \cdot \frac{(n-1)[(n-1)^2-1]}{[(n-1)-1][(n-1)^2+(n-1)+1]} \\ &= \frac{3}{2} \cdot \frac{(n-1)(n^2-2n)}{(n-2)(n^2-n+1)} \\ &= \frac{3}{2} \cdot \frac{(n-1)n}{n^2-n+1} \\ \text{then} \quad A_n &= A_{n-1} \cdot \frac{n^3+1}{n^3-1} \\ &= \frac{3}{2} \cdot \frac{(n-1)n}{n^2-n+1} \cdot \frac{n^3+1}{n^3-1} \\ &= \frac{3}{2} \cdot \frac{(n-1)n}{n^2-n+1} \cdot \frac{(n+1)(n^2-n+1)}{n^3-1} \\ &= \frac{3}{2} \cdot \frac{(n-1)n(n+1)}{n^3-1} \\ &= \frac{3}{2} \cdot \frac{n^3-n}{n^3-1}, \end{aligned}$$

so the claim follows by induction on n.

It now follows that

$$\begin{aligned} \lim_{n\to\infty} A_n &= \lim_{n\to\infty} \frac{3}{2} \cdot \frac{n^3-n}{n^3-1} \\ &= \frac{3}{2} \cdot \lim_{n\to\infty} \frac{n^3-n}{n^3-1} \\ &= \frac{3}{2} \cdot \lim_{n\to\infty} \frac{1-1/n^2}{1-1/n^3} \\ &= \frac{3}{2} \cdot 1 \\ &= \frac{3}{2}. \end{aligned}$$

Also solved by Kourosh Ghaderi, Hans de Moor, Grant Morgan, Winston Teitler, and Jan van Delden (the Netherlands)