Problem for 2012 January

Communicated by Dan Jurca

Suppose $A,\ B$, and C are three distinct points on an arbitrary parabola $\mathcal{P}$. Prove that the area of the triangle ΔABC is twice the area of the triangle bounded by the lines tangent to $\mathcal{P}$ at $A,\ B$, and C.

Solution by Dan Jurca

Without loss of generality we may suppose the parabola is the curve in the x-y plane defined by the equation $x^2 = 4py$, or $y = x^2/4p$. (This curve then consists of all points in the x-y plane equidistant from the focus at $(0, p)$ and the line—the directrix—with equation $y = -p$.) Then suppose the three points are at $A:\ (a, a^2/4p),\ B : (b, b^2/4p)$, and at $C : (c, c^2/4p)$. It follows that the area of the triangle with vertices at $A,\ B$, and C is the absolute value of the following determinant.

$$\begin{vmatrix} a & b & c \\ a^2/4p & b^2/4p & c^2/4p \\ 1 & 1 & 1 \end{vmatrix}$$

Equations of the lines tangent to the parabola at A and B are $y - a^2/4p = (a/2p)(x - a)$ and $y - b^2/4p = (b/2p)(x - b)$, respectively; and these line intersect at the following point.

$$\left(\frac{a+b}{2}, \frac{ab}{4p}\right)$$

The points of intersection of the other two pairs of tangent lines are similar; hence the area of the triangle bounded by the three tangent lines is the absolute value of the following determinant.

$$\begin{vmatrix} \dfrac{a+b}{2} & \dfrac{b+c}{2} & \dfrac{c+a}{2} \\ \dfrac{ab}{4p} & \dfrac{bc}{4p} & \dfrac{ca}{4p} \\ 1 & 1 & 1 \end{vmatrix}$$

By straightforward computation this determinant equals $-1/2$ times the determinant above.

Also solved by Lewis Felver and Kourosh Ghaderi